Christoffel-Darboux formula for orthogonal polynomials in several real variables

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Suppose that $\{p_n\}_{n=0}^{\infty}$ is a sequence of one variable real polynomials, which is orthogonal with respect to a Borel measure on $\mathbb{R}$. Then $\{p_n\}_{n=0}^{\infty}$ satisfies the three term recurrence relation, i.e.

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1}p_{n-1}(x), \quad n \geqslant 0,$$

with some $a_n, b_n \in \mathbb{R}$ (with $a_{-1} := 0$ and $p_{-1} := 0$). The Christoffel-Darboux formula is the equation:

$$\sum_{j=0}^{n} p_j(x)p_j(y) = a_n \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y}.$$

We are going to discuss a natural generalization of these formulas in the case of polynomials of several real variables. The three term recurrence relation is then the set of equations:

$$X_j Q_n \overset{V}{=} A_{n,j}Q_{n+1} + B_{n,j}Q_n + A_{n-1,j}^{\intercal}Q_{n-1}, \quad n \geqslant 0, \ j = 1, \ldots, d,$$

where $\{Q_k\}_{k=0}^{\infty}$ is a system of real orthogonal polynomials arranged in columns, where Q_k consists of polynomials of degree k; then $A_{n,j}$ and $B_{n,j}$ are real matrices of appropriate sizes. The notation "$\overset{V}{=}$" stands for "equality modulo an ideal V", which is inevitable, if we want to act in full generality (including e.g. polynomials orthogonal on a circle); this is a far-reaching refinement of results from [3, 4] published in [1]. The Christoffel-Darboux formula takes the form:

$$(x_j - y_j)\sum_{k=0}^{n} Q_k^{\intercal}(y)Q_k(x) \overset{V_2}{=} [A_{n,j}Q_{n+1}(y)]^{\intercal}Q_n(y) - Q_n(x)[A_{n,j}Q_{n+1}(y)]^{\intercal},$$

where $V_2 = V \otimes \mathcal{P}_d + \mathcal{P}_d \otimes V$ with $\mathcal{P}_d$ standing for the space of all polynomials in d variables (see [2]). Hopefully, the talk will be concluded with some examples (if time allows).

References

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