

## Accurate computation of the multidimensional fractional Laplacian

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In this talk we discuss approximation formulas for the fractional Laplacian  $(-\Delta)^{\alpha/2}$ ,  $0 < \alpha < 2$ , in the framework of the method approximate approximations. The fractional Laplacian appears in different fields of mathematics (PDE, harmonic analysis, semi-group theory, probabilistic theory) as well as in many applications (optimization, finance, materials science, water waves). If we introduce the convolution

$$\mathcal{N}_\alpha(f)(\mathbf{x}) = c_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{n-2+\alpha}} d\mathbf{y}, \quad c_{n,\alpha} = \frac{2^{\alpha-2} \Gamma(\frac{n-2+\alpha}{2})}{\pi^{n/2} \Gamma(\frac{2-\alpha}{2})}, \quad (1)$$

then the fractional Laplacian can be represented as the ordinary Laplacian of the volume potential  $\mathcal{N}_\alpha f$ ,

$$(-\Delta)^{\alpha/2} f(\mathbf{x}) = -\Delta \mathcal{N}_\alpha(f)(\mathbf{x}). \quad (2)$$

We propose a method of an arbitrary high order for the approximation of  $\mathcal{N}_\alpha f$  and  $(-\Delta)^{\alpha/2} f$ ,  $n \geq 3$ , which is based on the approximation of the function  $f$  via the basis functions introduced by approximate approximations (cf. [2]), which are product of Gaussians and special polynomials. Then the  $n$ -dimensional integral (1) applied to the basis functions is represented by means of a one-dimensional integral where the integrand has a separated representation, i.e., it is a product of functions depending only on one of the variables.

This construction enables to obtain one-dimensional integral representations with separated integrand also for the fractional Laplacian (2), when applied to the basis functions. An accurate quadrature rule and a separated representation of the density  $f$  provide a separated representation for  $\mathcal{N}_\alpha f$  and  $(-\Delta)^{\alpha/2} f$ . Thus, only one-dimensional operations are used and the resulting approximation procedure is fast and effective also in high-dimensional cases, and provides approximations of high order, up to a small saturation error. We prove error estimates and report on numerical results illustrating that our formulas are accurate and provide the predicted convergence rate 2, 4, 6, 8 (cf. [1]).

## References

- [1] F. Lanzara, V. Maz'ya, G. Schmidt, *Fast computation of the multidimensional fractional Laplacian*, *Applicable Analysis*, 2021.  
(<https://doi.org/10.1080/00036811.2021.1986025>)
- [2] V. Maz'ya, G. Schmidt, *Approximate approximations*, AMS, Providence (RI), 2007.