Discretization in Generalized Function Spaces

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The operation "discretization" usually means that functions are mapped to sequences of real or complex numbers. These sequences can moreover be finite or infinite. It is clear, "discretization" cannot be applied to all kinds of functions and it stands outside of two different kinds of spaces, function spaces and sequence spaces. Furthermore, four different Fourier transforms are involved, the integral Fourier transform for integrable (non-periodic) functions and the finite Fourier transform for periodic (locally integrable) functions, on one hand, and the Discrete-Time Fourier Transform (DTFT) and the Discrete Fourier Transform (DFT) for infinite and finite sequences, on the other hand [1]. However, "discretization" can be treated in Schwartz' generalized function spaces [2], such as the space of tempered distributions, where it is an operation that maps tempered distributions onto tempered distributions and its Fourier transform is the Fourier transform on tempered distributions. Recently it has been shown that this Fourier transform reduces to the four, usually defined Fourier transforms [3]. The setting of tempered distributions moreover allows to show that discretization and periodization are Fourier transforms of one another and their inverses, regularization and localization, form another Fourier transform pair [4]. A generalization of this concept is to understand discretization, periodization, regularization and localization as a family of four operations whose members are related to one another by three kinds of reciprocity, (i) reciprocity with respect to multiplication, (ii) reciprocity between multiplication and convolution and (iii) reciprocity with respect to convolution. Another important family is integration, differentiation, Fourier-domain integration and Fourier-domain differentiation. The former is Woodward's operational calculus [5, 6] and the latter is Heaviside's operational calculus [7]. Both are intensively used today in electrical engineering.

References

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