Real Paley-Wiener theorems

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A Paley-Wiener theorem is a characterization, by relating support to growth, of the image of a space of functions or distributions under a transform of Fourier type. This relation comes in terms of a compact and convex set in which the support of the function or distribution is included. In fact, the growth of $\hat{f}$ on $\mathbb{C}^d$ enables to retrieve the convex hull of the support of f, but no more precise information can be obtained from it. In the last years, a new type of results called "real Paley-Wiener type theorems" has received much attention. The idea is to try to bypass this theoretical obstruction for the classical Paley-Wiener theorems to "look inside" the convex hull of the support. The word "real" expresses that information about the support of f comes from growth rates associated to the function $\hat{f}$ on $\mathbb{R}^d$ rather than on $\mathbb{C}^d$ as in the classical "complex Paley-Wiener theorems". This theory was initiated by Bang, and here we follow the approach of Andersen and Andersen-De Jeu, facing the problem from the opposite point of view: starting by a rapidly decreasing function f we try to get information on the support of $\hat{f}$, which could be non-compact or even non-convex.
In particular, in [4] we work in the space $\mathcal{S}_\omega(\mathbb{R}^d)$ of rapidly decreasing ultradifferentiable functions for a weight ω (if $\omega(t) = \log(1+t)$ then $\mathcal{S}_\omega$ is the classical Schwartz space $\mathcal{S}$) and obtain the radius $R_{\hat{f}}$ of the support of $\hat{f}$ (which may be also $+\infty$) in terms of the derivatives of f or the Wigner transform of f:

$$R_{\hat{f}} = \lim_{n\to+\infty} \left(\max_{|\alpha|=n} \left\| e^{\lambda\omega\left(\frac{x}{|\alpha|+1}\right)} f^{(\alpha)}(x) \right\|_{L^p} \right)^{1/n}, \qquad \forall\lambda \geq 0, 1 \leq p \leq +\infty,$$

or

$$R_{\hat{f}} = \lim_{n\to+\infty} \| |\xi|_\infty^n \operatorname{Wig} f(x,\xi) \|_{L^{p,q}}^{1/n}, \qquad 1 \leq p, q \leq +\infty,$$

where $|\xi|_\infty = \max_{1\leq j\leq d} |\xi_j|$, and for the support of f:

$$R_f = \lim_{n\to+\infty} \| |x|_\infty^n \operatorname{Wig} f(x,\xi) \|_{L^{p,q}}^{1/n}, \qquad 1 \leq p, q \leq +\infty.$$

References

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