Modular-proximal operators for adaptive regularization

Luca Calatroni [a], **Claudio Estatico** [a], Marta Lazzaretti [a] [b]

[a] NRS, UCA, Inria, Lab. I3S (France)
[b] Department of Mathematics, University of Genova (Italy)
calatroni@i3s.unice.fr, estatico@dima.unige.it,lazzaretti@dima.unige.it

In this talk, structured optimisation problems of kind $\arg\min_{x\in X} f(x,y) + \lambda g(x)$ are considered as general form of Tikhonov-like functionals, where $f : X \times Y \longrightarrow \mathbb{R}$ represents a smooth convex fidelity term between the data $y \in Y$ and the solution $x \in X$, $g : X \longrightarrow \mathbb{R}$ is a proper, l.s.c., (possibly non-smooth) convex penalty term, and $\lambda > 0$ is the regularization parameter. In our approach, X and Y are both unusual variable exponent Lebesgue spaces $L^{p(\cdot)}$, that is, Lebesgue spaces where the exponent is not a constant value, but rather a function of the position of the domain [2, 3]. Due to their intrinsic space-variant geometrical properties, such Banach spaces can be naturally used for defining adaptive algorithms for the solution of ill-posed inverse problems.
For this purpose, we propose a proximal gradient algorithm in the (dual space of) $L^{p(\cdot)}$, where the proximal step is defined in terms of the modular function

$$\rho_{p(\cdot)}(x) := \int_\Omega \frac{1}{p(t)} |x(t)|^{p(t)} \mathrm{d}t,$$

which, thanks to its separability, allows for an efficient computation of the algorithmic forward-backward type iteration

$$x^{k+1} = \arg\min_{x\in L^{p(\cdot)}} \rho_{p(\cdot)}(x - x^k) + \lambda_k \langle \nabla f(x^k), x\rangle + \lambda_k g(x).$$

Convergence in function values is proved, with convergence rates depending on problem/space smoothness [5]. To show the effectiveness of the proposed modelling, some numerical tests highlighting the flexibility of the space $L^{p(\cdot)}$ are shown for exemplar signal and image deconvolution with mixed noise removal problems [5, 1, 4].

References

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