On the computation of fractional power of operators

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We consider the numerical approximation of $\mathcal{L}^{-\alpha}$, $0 < \alpha < 1$, where $\mathcal{L}$ is an accretive operator acting on a separable Hilbert space $\mathcal{H}$, with numerical range contained in a sector of the complex plane symmetric with respect to the real axis. This problem finds immediate application when solving equations involving a fractional diffusion term like $(-\Delta)^\alpha$ where Δ denotes the standard Laplacian. In this case the operator is self-adjoint.
By exploiting the existing representations of the function $\lambda^{-\alpha}$ in terms of contour integrals (see [1, 6]), after suitable changes of variable and quadrature rules one typically finds rational approximations of the type

$$\mathcal{L}^{-\alpha} \approx \mathcal{R}_{n-1,n}(\mathcal{L}), \quad \mathcal{R}_{n-1,n}(\lambda) = \frac{p_{n-1}(\lambda)}{q_n(\lambda)}, \quad p_{n-1} \in \Pi_{n-1},\, q_n \in \Pi_n,$$

where n is equal or closely related to the number of points of the quadrature formula. In this work we present a comparative analysis of the most reliable existing method based on quadrature rules, with particular attention to the error estimate and the asymptotic rate of convergence (see e.g. [2, 3, 4, 5]). The analysis is given in the infinite dimensional setting, so that all results can be directly applied to the discrete case, independently of the discretization used.

References

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