Explicit algebraic solutions, for low-degree polynomials, to Zolotarev's First Problem of 1868

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We report on our recent progress concerning the explicit algebraic solution of Zolotarev's First Problem (ZFP) of 1868 ([1], [2]), thus avoiding the application of elliptic functions. ZFP asks to determine, for $n \geq 4$ and $s > \tan^2\left(\frac{\pi}{2n}\right)$, the proper Zolotarev polynomial $Z_{n,s}$ which deviates least from zero in the uniform norm on $[-1,1]$ among all polynomials of form $x^n + (-ns)x^{n-1} + \dots$. By parametrization of algebraic curves, we have obtained in [4] a radical parametrization for $Z_{7,s}$ (currently the highest degree attacked by that method). Out of it, the solution of ZFP, for $n = 7$, can be recovered. In [3] we considered two alternative algebraic algorithms for explicitly solving ZFP (one was inspired by [5]). We now add a third one (inspired by [6]). Each algorithm creates a particular tentative form (depending on parameters α and β with $1 < \alpha < \beta$) of $Z_{n,s}$. To get the final form of $Z_{n,s}$ for the concretely chosen $n = n_0$ and $s = s_0$, the algorithms require as input compatible points $\alpha = \alpha_0$ and $\beta = \beta_0$ (depending on n_0 and s_0). In [3] we considered one variant how to determine α_0 and β_0. We now add two more variants. The variants involve Malyshev polynomials $F_n(\alpha)$ and $G_n(\beta)$, determinants with variable elements $d_{i,j}(\alpha,\beta)$, reduced relation curves $H_n(\alpha,\beta) = 0$, and function equations of form $s_n(\alpha,\beta) = s$. We show how to generate these terms by means of *Mathematica*-functions, e.g., GroebnerBasis. Pre-computed data to facilitate the computation of $Z_{n,s}$ and concrete examples, if $n \leq 13$, are provided and further existing non-elliptic approaches to ZFP are referenced.

References

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