Characterisation of the asymptotic behaviour of the mean–square of linear stochastic Volterra equations

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This talk concerns the asymptotic behaviour of autonomous stochastic Volterra equations of Itô–type. For simplicity, we will focus on scalar equations, but time permitting, finite dimensional equations will also be considered. The equation considered is

$$dX(t) = \int_{[0,t]} \nu(ds)X(t-s)\,dt + \int_{[0,t]} \mu(ds)X(t-s)\,dB(t), \quad t \geq 0; \quad X(0) = \xi, \qquad (1)$$

where ν and μ are scalar finite measures on $[0,\infty)$, B is a standard one–dimensional Brownian motion, and ξ is a random variable independent of B which has a finite second moment. In this situation, there is a unique continuous adapted process which obeys (1), which we call the solution. The finiteness of the second moment of $X(0)$ is inherited by $X(t)$ for all $t \geq 0$. This is the so–called mean square of the solution, denoted $\mathbb{E}[X^2(t)]$, and in applications the long–run behaviour (as $t \to \infty$) is of great interest. Indeed, many sufficient conditions are known under which the mean square tends to zero as $t \to \infty$ (so called mean–square asymptotic stability), using developments of techniques familiar in the deterministic theory, including Liapunov functionals, Razumikhin's technique and so on. However, a characterisation in terms of the problem data of the mean–square asymptotic stability (or L^1 stability in mean square) has not been known until now.

In this talk, we develop a linear deterministic convolution Volterra integral equation for a functional of the solution, and a representation of the mean square of X in terms of that functional. Thus, the analysis of the long–run behaviour of the mean square becomes a problem in deterministic integral equations. However, the problem data in the resulting integral equation (i.e. the kernel and forcing function) are given indirectly in terms of the data of equation (1) via the resolvent r of the underlying deterministic equation, namely

$$r'(t) = \int_{[0,t]} \nu(ds)r(t-s), \quad t \geq 0; \quad r(0) = 1, \qquad (2)$$

making the stability conditions hard to check.

Our main results show that it is possible (despite the fact that closed–form solutions of (2) are generally impossible to find) to give necessary and sufficient conditions on ν and μ such that the mean square tends to zero. Furthermore, the ideas involved allow stability characerisation results to be proven for perturbed equations of the form

$$dX(t) = \left(\int_{[0,t]} \nu(ds)X(t-s) + f(t)\right) dt + \left(\int_{[0,t]} \mu(ds)X(t-s) + g(t)\right) dB(t), \quad t \geq 0.$$

where f and g are deterministic functions. Some extensions of the arguments allow a characterisation of the exponential asymptotic behaviour in the mean square, which is perhaps the most widely studied type of weighted stability in applications. In that situation, we are even able to develop a stochastic characteristic equation in terms of ν and μ, which identifies the dominant Liapunov exponent of the solution.