Linear operators approximating discontinous functions. A probabilistic approach

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Many sequences of positive linear operators $\{L_n\}_{n\geq 1}$ allow for a probabilistic representation of the form $L_n(f,x) = Ef\left(x + \frac{Z_n(x)}{\sqrt{n}}\right), x \in I$, where I is a real interval, $f : I \to \mathbb{R}$ is any measurable function for which the preceding expectations exists, and $Z_n(x)$ is a random variable such that $x + Z_n(x)/\sqrt{n}$ takes values in I.
As an example, the classical Bernstein operators can be written as

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k} = Ef\left(x + \frac{Z_n(x)}{\sqrt{n}}\right), \quad x \in [0,1],$$

where $Z_n(x) = \frac{S_n(x)-nx}{\sqrt{n}}, x \in [0,1]$ and $S_n(x)$ is a random variable having the binomial distribution with parameters n and x. It is well known that for a bounded function f defined on $[0,1]$, $B_n(f,x) \to \frac{f(x-)+f(x+)}{2}$, where $f(x-)$ and $f(x+)$ are the left and right limits of f at x, respectively [3]. Rates of convergence in this case were provided by Bustamante et al.[2].
The main goal of this paper is to obtain linear operators acting on bounded functions on unbounded intervals such that $L_n(f,x) \to \alpha f(x+) + (1-\alpha) f(x-)$, with $0 \leq \alpha \leq 1$, giving at the same time rates of convergence. To do this, a probabilistic approach is used. As an illustration, we consider the sequence of linear operators $L_n^{(\alpha)} f(x) = Ef\left(x + \frac{Z_n^{(\alpha)}(x)}{\sqrt{n}}\right), x \in \mathbb{R}$, where $Z_n^{(\alpha)}(x)$ is a continuous random variable with probability density

$$\rho_\alpha(x) = \begin{cases} (1-\alpha)e^x & \text{if } x \leq 0 \\ \alpha e^{-x} & \text{if } x > 0 \end{cases} \qquad 0 \leq \alpha \leq 1.$$

References

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