Filtered product integration rules for the Hilbert Transform in $(-1,1)$

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We present a numerical method for approximating Hilbert transforms of the type

$$\mathcal{H}^u f(t) = \int_{-1}^{1} \frac{f(x)}{x-t} u(x)dx = \lim_{\epsilon\to 0} \int_{|x-t|\geq \epsilon} \frac{f(x)}{x-t} u(x)dx, \qquad -1 < t < 1,$$

where $u(x) = v^{\gamma,\delta}(x) := (1-x)^{\gamma}(1+x)^{\delta}$, $\gamma, \delta > -1$ is a Jacobi weight.
Fixed another Jacobi weight $w(x) = v^{\alpha,\beta}(x)$, and denoting by $\{p_j(w)\}_j$ the corresponding orthonormal polynomial sequence, fixed two integers $n,\ m$ with $0 < m < n$, the $n-$th filtered de la Vallée Poussin (VP) polynomial of f is defined as [1]:

$$V_n^m(w,f,x) = \sum_{k=1}^{n} f(x_k)\Phi_{n,k}^m(x),$$

where $\{\Phi_{n,k}^m\}_{k=1:n}$ are the fundamental VP polynomials

$$\Phi_{n,k}^m(x) = \lambda_{n,k}(w) \sum_{j=0}^{n+m-1} \mu_{n,j}^m p_j(w,x)p_j(w,x_k),$$

$\lambda_{n,k}(w)$ and x_k being the Cristhoffel numbers and the zeros of $p_n(w)$ respectively and $\mu_{n,j}^m$ are the following filters

$$\mu_{n,j}^m := \begin{cases} 1 & \text{if} \quad j = 0,\dots,n-m, \\ \dfrac{n+m-j}{2m} & \text{if} \quad n-m < j < n+m. \end{cases}$$

The new product type quadrature rule has been obtained by approximating f by $V_n^m(w,f)$ [2]. The convergence and stability are studied in suitable Besov type spaces. A comparison with the product quadrature rule based on the approximation of f by the Lagrange polynomial interpolating f at the same zeros $\{x_k\}_{k=1}^n$ of $p_n(w)$ is also proposed [3].

References

[1] W. Themistoclakis *Uniform approximation on* $[-1,1]$ *via discrete de la Vallée Poussin means*, Numerical Algorithms, Vol. 60, Issue 4 (2012), pp. 593–612.

[2] D. Occorsio, M.G. Russo, W. Themistoclakis *Filtered integration rules for finite weighted Hilbert transforms*, Journal of Computational and Applied Mathematics **410** (2022), 114–166.

[3] M.R. Capobianco, G. Mastroianni, M.G. Russo, *Pointwise and uniform approximation of the finite Hilbert transform,* in: Approximation and Optimization, Vol. 1, Transilvania, Cluj Napoca, (1997) 45–66.