FAATNA20>22

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INTERNATIONAL CONFERENCE ON

FUNCTIONAL ANALYSIS, APPROXIMATION THEORY AND NUMERICAL ANALYSIS

FAATNA 20>22

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- **Francesco Altomare**, *University of Bari Aldo Moro (Italy)*
- **José Bonet**, *Universitat Politècnica de València (Spain)*
- **Len Bos**, *University of Verona (Italy)*
- **Francisco Marcellán-Español**, *Universidad "Carlos III" de Madrid (Spain)*
- **Hrushikesh Mhaskar**, *Claremont Graduate University (USA)*
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- **Gianluca Vinti**, *University of Perugia (Italy)*
- **Yuesheng Xu**, *Old Dominion University (USA)*

Contents

Plenary Speakers

Some new results about the convergence of sequences of positive linear operators and functionals

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The talk will be devoted to report some recent results which have been established in [1] and [2] and which are concerned with the study of the limit behaviour of the sequences of the positive linear functionals and operators associated with integrated generalized means defined with respect to a given probability Borel measure on a Borel convex subset of a Hilbert space.

The main results are easily achieved through some new Korovkin-type theorems for composition operators and for functionals which are established in the context of function spaces defined on a metric space.

Several applications are showed in the special cases of bounded and unbounded real intervals which involve the most common integrated means. Furthermore, some consequences concerning the convergence in distribution, and hence stochastically, of generalized means of vector-valued random variables are also presented.

The final part of the talk will be addressed to discuss some applications to the so-called box integral problem, i.e., the problem to evaluate the limit behaviour as $n \to \infty$ of the average distance between two points of $[0,1]^n$ randomly chosen according to a given distribution on $[0,1]^n$ ([3]).

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The Fréchet algebra of uniformly convergent Dirichlet series

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In the first part of this lecture we recall several results about Bohr's problem concerning the largest possible strip on which a Dirichlet series of complex numbers converges uniformly but not absolutely. We will mention classical work by Harald Bohr (1913), Bohnenblust and Hille (1931), and recent one by Boas, Defant, Frerick, García, Khavinson, Maestre, Ortega-Cerdá, Ounaïes and Seip, among others.

In the second part we report about our work. Motivated by a classical result of Bohr that the abscissa of boundedness and the abscissa of uniform convergence coincide for a Dirichlet series and by an improved Montel principle due to Bayart in 2002, we investigate the Fréchet algebra of all Dirichlet series that are uniformly convergent in all the half-planes of complex numbers with positive real part. When endowed with its natural metrizable locally convex topology, this space is complete, Schwartz, not nuclear, has a Schauder basis and contains isomorphically the space of analytic functions on the open unit polydisc. The behaviour of composition operators, and the operators of differentiation and integration in this space is also investigated.

Optimal Prediction Measures

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Suppose that $K \subset \mathbb{C}^d$ is a compact set. For data given on K it is possible, by means of polynomial regression, to *predict* (or extrapolate) a value at a point z_0 *exterior* to K. An optimal prediction measure is the probabilty measure on K which describes the data distribution on K for which the predicted value has least variance.

We will discuss this problem and its relation to another classical approximation problem, give some examples, and discuss some conjectures and open problems.

Coherent pairs of linear functionals and Sobolev Orthogonal Polynomials

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The study of polynomials orthogonal with respect to a Sobolev inner product has attracted the attention of many researchers during the last years (see [3] for a survey on this topic). Their constructive approach in the univariate and multivariate cases, respectively, as well as their properties in the framework of approximation theory (Fourier series) and numerical analysis (spectral methods for Boundary Value problems for ODEs and PDEs) have been deeply analyzed.

In this talk we will focus the attention on the Sobolev orthogonal polynomials associated with the so-called coherent pairs of measures ([2]) and the coherent pairs of the second kind (see [1]), respectively. The characterization of such pairs of measures will be discussed. In the second case, when one of the measures is classical (Jacobi and Laguerre) we will analyze the corresponding sequences of Sobolev orthogonal polynomials. We will deduce analytic properties of them.

Coherent pairs and coherent pairs of the second kind of Borel measures supported on the unit circle will be also presented (see [4]). Some open problems will be discussed.

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Localized kernel method in signal processing and machine learning

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Many applications in signal processing and machine learning requires the use of intrinsically global data for obtaining local analysis of the objects involved. For example, the problem of separation of stationary signals can be formulated as recuperation of a finitely supported measure on the complex unit circle using finitely many Fourier coefficients of the measure. One way to mitigate the curse of dimensionality in machine learning is to apply local learning based on a random sample taken from an unknown probability distribution. For example, assuming the data lies on a compact metric measure space, we may wish to develop a network so that different subnetworks are responsible for the data on different parts of the metric space. Motivated by such applications, we have developed a family of localized kernels based on different global orthogonal systems in various settings: the complex unit circle, Jacobi and spherical polynomials, multivariate Hermite polynomials, manifolds etc. We state a Tauberian theorem that gives a general construction for such kernels, and illustrate the use of the localized kernels in various theoretical and practical settings.

Generalized Quadrature Formulas of Gaussian Type

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This lecture is devoted to quadrature processes of Gaussian type (cf. [1]). Beside the basic facts on the weighted Gaussian formulas on the real line and several generalizations and modifications, the connections with orthogonal polynomials and the basic procedures for their numerical and symbolic generation for arbitrary measures, including available software, are presented. Several examples with non-classical weight functions and their applications in summation of slowly convergent series, as well as to computation of some special integrals and functions are given. If the information data $\{f(x_k)\}_{k=1}^n$ in the standard n-point Gaussian quadrature formula is replaced by $\{(\mathcal{A}^{h_k}f)(x_k)\}_{k=1}^n$, where $\mathcal{A}^h$ is an extension of some linear operator $\mathcal{A}^h : \mathbb{P} \to \mathbb{P}$ ($h \geq 0$, $\mathbb{P}$ is a space of all algebraic polynomials), we get a non-standard quadrature formula [2, 3]. Typical linear operators can be an average (Steklov's) operator, some difference or differential operators. One type of these formulas based on values of certain linear differential operators at some nodes (see [4]) can be interesting in applications when the operator values are available, instead of the values of the original integrand function.

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Positive operators, inequalities, and stochastic convex orders

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Joint work with Ana Maria Acu

Let $p_{n,j}(x) := \binom{n}{j} x^j (1-x)^{n-j},\ x \in [0,1],\ 0 \le j \le n$. The analytic inequality

$$\sum_{i=0}^{n} \sum_{j=0}^{n} \left[p_{n,i}(x)p_{n,j}(x) + p_{n,i}(y)p_{n,j}(y) - 2p_{n,i}(x)p_{n,j}(y) \right] f\left(\frac{i+j}{2n}\right) \ge 0,$$

valid for each convex function $f \in C[0,1]$, is the simplest illustration of the results presented in this talk. It is related with the shape preserving properties of the Bernstein-Schnabl operators, see [4, Sec. 3.4]. Its first proof [6] uses stochastic convex orderings. The first *analytic* proof [1] was followed by many other proofs, in analytic or probabilistic terms, involving more general families of operators and convex functions of higher order, see [2], [5] and the references therein. The talk surveys the existing results in this area and presents some new, very recent results and problems [3].

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A mathematical model for the reconstruction and the enhancement of digital images and its applications in the medical field

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A mathematical model based on the theory of sampling type operators will be described, together with the various approximation properties ([1, 4, 3, 5]). Later, it will be shown how the implementation of the mathematical results led to the study of a problem in the medical field ([2]).

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Sparse Machine Learning in Banach Spaces

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We will discuss recent developments in sparse machine learning methods in Banach spaces. To motivate this important topic, we begin with a review of the classical classification problem. In particular, we will present new understanding in representer theorems of machine learning methods in Banach spaces.

Special Sessions

S1. Integral Equations: recent developments in numerics and applications

The aim of this special session is to present recent mathematical and computational developments on integral equations, as well as their applications. In particular, the talks will focus on integral equations with nonsmooth data defined on intervals, and on boundary integral equations associated with stationary or time-dependent PDE boundary value problems. Recent and efficient numerical approaches will be presented, together with their applications to different fields such as for example, acoustics, electromagnetics, heat conduction, elastodynamics.

Organizers:

Luisa Fermo, University of Cagliari

Letizia Scuderi, Politecnico di Torino

Fast Energetic BEM for time-domain acoustic and elastic 2D scattering problems

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We consider acoustic and elastic wave propagation problems in 2D unbounded domains, reformulated in terms of space-time Boundary Integral Equations (BIEs). The BIEs are set in a weak form related to the energy of the system and then discretized by a Galerkin-type Boundary Element Method (BEM): this approach, called Energetic BEM, has revealed accurate and stable even on large time intervals of analysis [1, 2].
The Energetic BEM matrices have a Toeplitz lower triangular block structure, where blocks, generated using standard Lagrangian piecewise polynomial space basis functions, become fully populated for growing time; hence the overall memory cost of the energetic BEM is $O(M^2N)$, M and N being the number of the space degrees of freedom and the total number of performed time steps, respectively. This can be a drawback for the application of such method to large scale problems. To overcome this issue, we have proposed in [3] a fast technique based on the Adaptive Cross Approximation (ACA). Indeed, the core of this procedure is the approximation of sufficiently large time blocks of the energetic BEM matrix through the partially pivoted ACA algorithm introduced in [4], which allows to compute only few of the original entries. This leads to reduced assembly time, which for the energetic BEM is generally relevant, coupled with reduced memory storage requirements. Additionally, the consequent acceleration of the matrix/vector multiplication together with a marching on time procedure, leads to remarkable reduction of the computational solution time. The effectiveness of the proposed method is theoretically proved and several numerical results are presented and discussed, with some further advancements.

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Filtered interpolation and numerical resolution of systems of hypersingular integro-differential equations

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In this talk we show how a collocation-quadrature method based on de la Vallée Poussin filtered interpolation at Chebyshev nodes can be applied for solving systems of hypersingular integro-differential equations (HIDE) of the following type

$$\begin{aligned}
\sigma\zeta_1(y) \quad + \quad & a\zeta_2'(y) + \frac{b}{\pi}\int_{-1}^{1}\frac{\zeta_2(x)}{(x-y)^2}dx - \frac{1}{\pi}\int_{-1}^{1}\kappa_1(x,y)\zeta_2(x)dx = g_1(y), \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad y \in (-1,1), \\
\sigma\zeta_2(y) \quad + \quad & a\zeta_1'(y) + \frac{b}{\pi}\int_{-1}^{1}\frac{\zeta_1(x)}{(x-y)^2}dx + \frac{1}{\pi}\int_{-1}^{1}\kappa_2(x,y)\zeta_1(x)dx = g_2(y),
\end{aligned}$$

with $\sigma \in \mathbb{R}$, $\kappa_i(x,y)$ and $g_i(y), i = 1,2$, given functions, the constants $a, b \in \mathbb{R}$ s.t. $a^2 + b^2 = 1$, and the unknown solution $\mathcal{Z} = (\zeta_1, \zeta_2)$ a differentiable function, satisfying the zero boundary condition

$$\mathcal{Z}(-1) = \mathcal{Z}(1) = 0.$$

The above systems are of interest because, for example, appear in the model describing the weak interface between two elastic materials containing a periodic array of micro-crazes [3]. Indeed, the boundary conditions yielding there to the solution of the posed problem are given in terms of an HIDE system.

The method is based on the procedure proposed in [2]. In the special case $\kappa_1 \equiv \kappa_2$ it is conveniently combined with a procedure presented in [1] that converts the system into a separable system of two independent equations. We prove its stability and uniform convergence in Hölder-Zygmund spaces of locally continuous functions and we show its efficiency through some numerical examples.

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Energetic Boundary Element Method for 3D wavefield modelling

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We consider elastodynamic (vector) wave equation, defined in unbounded domains external to 3D bounded ones, endowed with null initial conditions and with a Dirichlet condition on the boundary. For its numerical solution, we reformulate the original differential problem in terms of a space-time Boundary Integral Equation (BIE) and then we employ a weak formulation linked to the energy of the system in order to achieve, in the approximation phase by means of a Galerkin-type Boundary Element Method (BEM), accurate and stable numerical results. This approach, called Energetic BEM (EBEM), leads to a linear system whose matrix has Toeplitz lower-triangular block structure, that allows the acceleration of the solution phase. As a direct consequence of the flexibility of the EBEM, a large body of literature has risen after the pioneering paper [1], to witness its capabilities to simulate 3D acoustic [2] and 2D elastodynamic [3] wave propagation in semi-infinite or infinite media. However, the extension of the EBEM to 3D elastic problems is not straightforward, since the energetic full space-time discretization requires double integration both in space and in time. Since a key ingredient for the success of the EBEM is the efficient and accurate evaluation of all the involved integrals, the selected formulation could be quite challenging in large scale applications. Nevertheless, if standard (constant) time and shape functions are employed, the double integration in time can be performed analytically and one is left with the task of evaluating double space integrals, whose integration domains are generally delimited by the wavefronts of the primary and the secondary waves. In order to exactly detect this latter, and consequently to preserve the stability properties of the EBEM, we choose boundary meshes made by triangular elements with straight sides and we propose an ad-hoc numerical integration scheme, tailored for the correct domain of integration.

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E-BEM for the resolution of 2D interior elastodynamic problems

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The study of elastic wave propagation phenomena can find relevant applications in many fields such as mechanical engineering, physics and geological soil analysis, giving pulse, for the accurate resolution of the inherent differential problem, to the study and the implementation of suitable numerical strategies. Among them, the Boundary Element Method (BEM) stands out as a powerful numerical tool, whose implementation requires to rewrite the physical displacement or traction as unknown of a Boundary Integral Equation (BIE) defined just at the boundary of the propagation domain, leading to a dimensional reduction of the initial problem. The BIE can be reformulated through energy arguments into a weak form, that we numerically solve after a space-time discretization of Galerkin type (E-BEM). This energetic method has been successfully implemented for the analysis of scalar wave propagation problems [2], while recent advances in 2D exterior elastodynamic soft scatterings can be found in [1, 3], where several numerical results are reported as validation of the E-BEM long time stability properties. In this contribution we will extend the outcomes obtained in the elastodynamic framework, with the purpose of treating Neumann and mixed boundary condition problems for the approximation of the resultant displacement in interior bounded domains. We will provide an overview of the possible combinations of the energetic weak formulations, that, depending on specific integral operators with kernels defined as spatial derivatives of the 2D elastodynamic Green tensor, allows us to incorporate the boundary conditions in the weak problem. Algorithmic considerations about the discretization of the obtained weak formulations will be included in the contribution, with a focus on the quadrature strategies required for the computation of matrix entries featured by different type of spatial singularities. In the end, we will present numerical tests concerning the implementation of the method, highlighting its efficiency and the long-time stability of the approximated solutions obtained solving by E-BEM for the considered elastodynamic problems.

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An averaged Nyström method for 2D Fredholm integral equations

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This talk deals with the numerical solution of the following equation defined on the square $\mathcal{S} := [-1,1] \times [-1,1]$

$$(I - K)f = g,$$

where f is the bivariate function to be recovered, I is the identity operator, g is a given right-hand side sufficiently smooth on $(-1,1) \times (-1,1)$ possibly with algebraic singularities on the boundary, and Kf is the integral operator. It is defined as

$$(Kf)(\mathbf{y}) = \int_{\mathcal{S}} k(\mathbf{x},\mathbf{y}) f(\mathbf{x}) w(\mathbf{x}) d\mathbf{x}, \quad \mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathcal{S},$$

where k is a known kernel function defined on $\mathcal{S} \times \mathcal{S}$ and

$$w(\mathbf{x}) = w_1(x_1) w_2(x_2),$$

with

$$w_i(x_i) = (1 - x_i)^{\alpha_i} (1 + x_i)^{\beta_i} \quad \alpha_i, \beta_i > -1, \quad i = 1, 2.$$

The function w is the product of two classical Jacobi weights and then contains any algebraic singularities on the boundaries of the kernel.

A Nystrom-type method based on Gauss and anti-Gauss cubature rules is developed and an averaged Nyström interpolant is proposed to improve the accuracy of the solution.

Numerical tests show the performance of the approach.

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Coupling of curved virtual element with boundary element methods for exterior wave propagation problems

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We consider the wave equation defined on the exterior of a bounded 2D space domain, endowed with a Dirichlet condition on the boundary. We propose a numerical method that approximates the solution using computations only in an interior finite domain. This is obtained by introducing a curved smooth artificial boundary on which a non-reflecting boundary condition, defined by a boundary integral equation, is imposed. The approach we consider allows for solving the original problem by means of the coupling of an interior domain method with a boundary element one associated with the artificial boundary. For the space discretization in the interior computational domain, we propose a Galerkin approach based on the Curvilinear Virtual Element Method (CVEM), and for the time discretization we use the classical Crank-Nicolson method. For the approximation of the non-reflecting condition on the artificial boundary, we apply a standard collocation Bound- ary Element Method (BEM) combined with a Lubich time convolution quadrature formula. Some numerical results are presented to test the performance of the proposed approach and to highlight its effectiveness.

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Recent results on the stability of the non-symmetric coupling of finite and boundary elements

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We consider the non-symmetric coupling of finite and boundary elements to solve second order uniform elliptic partial differential equations defined in unbounded domains. We present a novel condition that ensures the ellipticity of the associated bilinear form, keeping track of its dependence on the linear combination coefficients of the interior domain equation with the boundary integral one. We show that an optimal ellipticity condition, relating the minimal eigenvalue of the diffusion matrix to the contraction constant of the shifted double-layer integral operator, is guaranteed by choosing a particular linear combination. This latter condition is always satisfied when the interface is a circle. These results generalize those obtained in Of and Steinbach [2] and [3], and in Steinbach [4] where the simple sum of the two coupling equations has been considered. Numerical examples confirm the theoretical results on the sharpness of the presented estimates.

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A stable BIE method for Laplace's equation with Neumann boundary conditions in domains with piecewise smooth boundaries

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It is well known that, in classical potential theory, the Laplace equation with Neumann boundary conditions can be reduced to integral equations of the second kind defined on the boundary of the domain. In particular, in this talk we consider the exterior Neumann problem in a open bounded simply connected planar domain $\Omega \subset \mathbb{R}^2$ with a piecewise smooth boundary Γ

$$\begin{aligned} &\Delta u(x) = 0, && x \in \mathbb{R}^2 \setminus \bar{\Omega}, \\ &\frac{\partial u(x)}{\partial \nu_x} = f, && x \in \Gamma, \\ &|u(x)| = o(1), && \text{as } |x| \to \infty, \end{aligned}$$

where ν_x denotes the outward-pointing unit normal vector to Γ at x.

Using the single layer representation of the potential

$$u(x) = -\int_\Gamma \psi(y) \log|x-y| dS(y), \quad x \in \mathbb{R}^2 \setminus \bar{\Omega},$$

the differential problem is reformulated in terms of the boundary integral equation (BIE)

$$-\pi\psi(x) - \int_\Gamma \frac{\partial}{\partial \nu_x} \log|x-y|\psi(y) dS(y) = f(x), \qquad x \in \Gamma,$$

whose solution ψ is the single layer density function and has singularities near the corners of the boundary.

A Nyström type method based on a proper Gauss-Jacobi-Lobatto quadrature formula is proposed for its approximation. Taking into account the known behavior of the solution, the analysis is carried out in proper weighted spaces of continuous functions.

Introducing a modification of the method in the vicinity of the corners we are able to prove both convergence and stability in such spaces.

The efficiency of the method is shown by illustrating some numerical tests.

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New fast and oblivious convolution quadrature based on the global inversion of the Laplace transform

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A new generation of fast algorithms with reduced memory requirements and easy to implement for the numerical approximation of Volterra type convolutions is presented. The main ingredient is the global in time inversion of the Laplace transform of the convolution kernel. In several important applications, the Laplace transform of the kernel, also called *transfer operator*, is such that the approximation of the inverse mapping in an interval $[\delta, T]$, with $T \gg \delta$, can be computed with a unique set of quadrature weights and nodes. In order to develop such efficient quadratures we need to use all the information about the problem and thus focus on families of applications, such as fractional integrals and derivatives [1], integral formulations of Shrödinger problems [2] and evolutionary problems governed by sectorial operators [3]. Numerical results supporting the theory will be presented, showing the advantages and the potential of this new approach.

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A numerical method for Volterra integral equations based on equispaced nodes

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The talk deals with the numerical treatment of Volterra integral equations of the type

$$f(s) + \mu \int_0^s k(t,s) f(t) (t-s)^\alpha t^\beta dt = g(s), \quad s \in (0,1),$$

where f is the unknown function, k and g are given functions, $\mu \in \mathbb{R}$, and $\alpha, \beta \geq 0$.

Here we propose a Nyström method based on a quadrature formula whose knots are equally spaced points of $[0,1]$. The use of equidistant points is crucial in many engineering and mathematical physics problems which are modeled by Volterra equations. Indeed, very often the functions k and g are available only at a discrete set of equispaced nodes, e.g. as results of experiments or measurement on the field. In these cases the classical methods based on piecewise polynomial approximation offer a lower degree of accuracy, while the efficient procedures based on the zeros of orthogonal polynomials cannot be used.

The quadrature formula herein is obtained by means of the sequence $\{B_{m,\ell}(f)\}_m$ of the so-called Generalized Bernstein polynomials, where $B_{m,\ell}(f)$ is the ℓ iterated boolean sum of the classical Bernstein polynomial $B_m(f)$, i.e.

$$B_{m,\ell}(f) = f - (f - B_m(f))^\ell, \qquad \ell \in \mathbb{N}, \quad B_{m,1}(f) = B_m(f).$$

We note that $B_{m,\ell}(f)$ requires the samples of f at equispaced nodes as well as the "original" Bernstein polynomial $B_m(f)$. However, differently from $B_m(f)$, its rate of convergence improves as the smoothness of the function increases. Indeed, by approximating $f \in C^{2\ell}([0,1])$ by $\{B_{m,\ell}(f)\}_{m,\ell}$, it is $\|f - B_{m,\ell}(f)\|_\infty \sim \mathcal{O}(m^{-\ell})$.

Finally, stability and convergence of the Nyström method are proved in Hölder–Zygmund type spaces and some numerical tests are given to confirm the theoretical estimates.

“Truncated” Gaussian quadrature rules for exponential weights

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In 1987 P. Barrucand described to G. Mastroianni and G. Monegato some numerical experiments showing that the following “truncated” Gaussian rule

$$\int_0^\infty f(x)\,\mathrm{e}^{-x}\,\mathrm{d}x = \sum_{i=1}^{n} \lambda_{m,i} f(x_{m,i}) + R_{m,n}(f)\,, \qquad m > n\,,$$

where $x_{m,i}$ are the zeros of the mth Laguerre polynomial and $\lambda_{m,i}$ are the associated Christoffel numbers, provides better approximations than those obtained using the ordinary Gauss–Laguerre formula. Subsequently, G. Mastroianni and G. Monegato studied the convergence of the “truncated” Gauss–Laguerre rule and the related Nyström method for Fredholm integral equations [3, 4].

In the past twenty years this procedure has been extended in several directions and the underlying theory has been deepened (see [2, 1] and the references therein).

In this talk we will discuss new results and numerical advantages of “truncated” quadrature rules related to different exponential weights on bounded or unbounded intervals of the real line.

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Sinc-collocation methods with consistent collocation points for Fredholm integral equations of the second kind

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This talk is concerned with Fredholm integral equations of the second kind of the form

$$u(t) - \int_a^b k(t,s)u(s)\mathrm{d}s = g(t), \quad a \le t \le b,$$

where k and g are given continuous functions, and u is the solution to be determined. Most numerical methods in literature do not work well when the functions k and g have derivative singularity at the endpoints. To overcome the difficulty, Rashidinia–Zarebnia [2] proposed a Sinc-collocation method, which was derived without assuming differentiability at the endpoints. Okayama et al. [1] reformed their method and prove that it can attain $\mathrm{O}(\exp(-c\sqrt{N}))$, where N (approximately) denotes half the number of collocation points. The method is based on the Sinc approximation combined with the SE transformation: $t = \psi_1(x) = (b-a)\tanh(x/2)/2 + (b+a)/2$. They further improved the method by replacing SE transformation with the DE transformation: $t = \psi_2(x) = \psi_1(\pi \sinh x)$, and proved that the improved method can attain higher convergence rate: $\mathrm{O}(\exp(-cN/\log N))$.

However, those methods are a bit stressful to implement, especially in the case of system of Fredholm integral equations. This is because $(2N+1)$ collocation points t_i are taken as

$$t_i = \begin{cases} a & (i = -N) \\ \psi_\ell(ih) & (i = -N+1, \ldots, N-1) \\ b & (i = N) \end{cases}$$

and not consistent at $i = -N$ and $i = N$. To remedy this issue, this study derives another Sinc-collocation methods with the consistent collocation points

$$t_i = \psi_\ell(ih) \quad (i = -N, \ldots, N).$$

Furthermore, this study proves that the new method with $t = \psi_1(x)$ can attain $\mathrm{O}(\exp(-c\sqrt{N}))$, and the new method with $t = \psi_2(x)$ can attain $\mathrm{O}(\exp(-cN/\log N))$. Numerical comparisons are also presented in the talk.

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Numerical Dynamics of Integrodifference Equations: Invariant Manifolds

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Integrodifference equations are popular tools in theoretical ecology to model the spatial dispersal of populations evolving with nonoverlapping generations. Their behavior is often illustrated using numerical simulations based on various discretization methods.

In this talk, we discuss the persistence and behavior of their invariant manifolds under spatial discretization and establish convergence preserving the order of the numerical method. Our approach ranges from classical stable and unstable manifolds of autonomous equations to the full hierarchy of invariant fiber bundles for nonautonomous problems. Moreover, various ambient state spaces are discussed.

Regularized minimal-norm solution of overdetermined first kind integral models

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First kind integral equations arise in many mathematical models. A typical situation is when one needs to identify certain parameters of a physical system confined in a specified domain, from observations detected outside the domain by, e.g., mechanical or electromagnetic waves. Some sensing devices used in such settings allow different configurations, leading to overdetermined systems of integral equation with discrete data of the form

$$\int_a^b k_\ell(x_{\ell,i},t)\,f(t)\,dt = g_\ell(x_{\ell,i}), \qquad \ell=1,\ldots,m, \quad i=1,\ldots,n_\ell,$$

Such problems are typically ill-posed: they admit infinitely many solutions and, because of experimental errors, must be solved in the least-squares sense.

We will describe a numerical method for computing the minimal-norm solution of the problem in the presence of boundary constraints, which stems from the Riesz representation theorem and the theory of reproducing kernel Hilbert spaces (RKHS) [1, 2]. The algorithm involves the singular system of the integral operator associated to the overdetermined system and naturally induces a regularization technique. Numerical experiments, both synthetic and deriving from an application in applied geophysics, show that the new method is extremely effective when the sought solution is smooth, but produces significant information even for non-smooth solutions.

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Numerical method for BVP problems on real line

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We consider the following Boundary Value Problems (BVPs) of second order on the real axis

$$\begin{cases} f''(x) - \mu a(x) f(x) = h(x), \\ f(-\infty) = f(+\infty) = 0, \end{cases} \tag{1}$$

where $\mu \in \mathbb{R}$, a and h are given functions satisfying suitable assumptions and f is the unknown solution. Boundary-value problems on infinite intervals are of interests because are model for many problems arising from physical phenomena, such as the flow of a gas through a semi-infinite porous medium or non-Newtonian fluid flows.

Following an argument in [1], we reduce the problem (1) to the following equivalent Frendolm integral equation

$$f(t) - \int_{\mathbb{R}} G(x,t) a^*(x) f(x)\, dx = \int_{\mathbb{R}} G(x,t) h(x)\, dx, \tag{2}$$

where

$$G(x,t) = -\frac{1}{2} \begin{cases} e^{-t} e^{x}, & -\infty < x \le t, \\ e^{t} e^{-x}, & t \le x < \infty, \end{cases}$$

is the Green's function and $a^*(x) = (\mu a(x) - 1)$. We propose to solve the above integral equation by a Nyström type method based on a product quadrature rule. The stability and convergence of the method are studied. Numerical tests are shown.

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S2. Meshless Approximation Methods: new perspective and applications

Over the past twenty years, meshless methods became one of the major interest field in approximation theory. This session focuses in the recent development and improvement of existent meshless techniques, as well as in the presentation of new approaches and applications.

Organizers:

Francesco Dell'Accio, University of Calabria
Alessandra De Rossi, University of Turin
Elisa Francomano, University of Palermo

Adaptive Residual Sub-sampling Methods for Kernel-based Interpolation

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In this talk we present an improved version of the residual sub-sampling method in [1] for adaptive interpolation by radial kernels. We introduce in the context of sub-sampling methods a maximum profile likelihood estimation criterion for the optimal choice of the kernel shape parameter [2]. This selection is completely automatic, provides reliable and accurate results for any kernel, and, unlike the original residual sub-sampling method, guarantees that the kernel interpolant exists uniquely. The performance of this new interpolation scheme is tested by numerical experiments on one and two dimensional test functions.

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Physics-informed neural networks approach for solving Gray-Scott systems

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A physics-informed neural network (PINN) is employed to approximate the solution of nonlinear partial differential equation systems. In this talk, we present an approach for solving different configurations for the Gray-Scott, a reaction-diffusion system that involves an irreversible chemical reaction between two reactants. Computational results show that the PINN can successfully provide an approximated solution in a variety of conditions, also reproducing the characteristic Turing patterns in the unstable region of the model's parameter space, through a supervised approach that relies on a finite difference method (FDM).

Joint work with M.O.D.A.L Laboratory

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A Kriging Estimator based on the Partition of Unity Method

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We propose a new tool, namely the Kriging Estimator [3], based on the Partition of Unity method (KEPU). The theoretical studies about the propagation of the uncertainties from the local predictors to the global one enable us to define the PU method (see [1, 2, 4]) in a stochastic framework. As numerically confirmed, when the number of instances grows, such a method allows us to significantly reduce the usually high complexity cost of Gaussian processes.

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Smooth approximation and interpolation of scattered data on the sphere with linear precision by quadrangulations

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The problem of interpolation on the sphere arises in the study of many physical phenomena, such as temperature, rainfall, pressure, ozone distribution or gravitational forces, measured at scattered points on the earth surface. A comprehensive survey on various approaches to solve this problem has been provided by G.E. Fasshauer and L.L. Schumaker in 1998 [5]. Later, in 2010, R. Cavoretto and A. De Rossi proposed an approach which involves a modified spherical Shepard's interpolant and zonal basis functions as local approximants [1]. In line with previous studies [2, 3, 4], in this talk we discuss a new approach based on quadrangular Shepard basis functions on the sphere combined with linear interpolants at quadrangulations of the scattered points. In particular, the basis functions are the normalization of the product of the inverse geodesic distance to the vertices of the quadrangulation while the linear interpolants are defined by spherical polynomials [6]. The resulting operator reproduces linear polynomials on the sphere and interpolates the given data. Numerical experiments on variuos sets of scattered points demonstrate the effectiveness of the approximation.

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Exploring the expressive power of ExSpliNet, a new spline-based neural network model

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The approximation of high-dimensional functions is an extremely challenging task. Classical methods, based on a mesh, suffer in general from the so-called *curse of dimensionality* and thus in practice they are only suited for addressing lower-dimensional problems. On the other hand, machine learning techniques that are mesh-free, and in particular *(deep) neural networks*, seem to be able to overcome this issue for different kinds of high-dimensional problems, especially in the context of image analysis and pattern recognition.

The *ExSpliNet model*, introduced in [1], is an interpretable and expressive neural network model that combines ideas of Kolmogorov neural networks, ensembles of probabilistic trees, and multivariate B-spline representations. The model, inspired by Kolmogorov's Superposition Theorem, uses univariate splines as inner functions that feed L-variate tensor-product splines as outer functions, all of them represented in terms of B-splines. Here, L is supposed to be not too high. The inner functions act as low-dimensional feature extractors, while the outer functions can be regarded as probabilistic trees. The ExSpliNet model can be efficiently evaluated thanks to the computational properties of B-splines. In [1], the effectiveness of the proposed model was also tested on classical machine learning benchmark datasets and it was numerically illustrated that the model is particularly suited for data-driven (smooth) function approximation and to face differential problems, in the spirit of physics-informed neural networks.

In this talk, after discussing the ExSpliNet model's definition and its main properties, we focus on the approximation capabilities and present two constructive results that mitigate the curse of dimensionality. More precisely, following ideas similar to the ones proposed in [2, 3], we obtain error bounds for the ExSpliNet approximation of a subset of multivariate continuous functions and also of multivariate generalized bandlimited functions. The curse of dimensionality is lessened in the first case, while it is completely overcome in the second case.

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Moving Least Square Approximation using Variably Scaled Discontinuous Weight Functions

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The moving least square (MLS) is an approximation (of low order) method introduced by Shepard [4] and generalized to higher approximation order by Bos and Salkauskas [1]. The MLS method approximates a function given at irregularly spaced points by weighted least square approximations. The smoothness of the MLS approximant is decided by the smoothness of the weight functions (cf. [5]). Moreover, the weight function are considered to be smooth functions of some order, regardless of the smoothness of the underlying function to be approximated. However, in case that the underlying functions possess some discontinuities at some points, smooth approximants become highly oscillatory near the discontinuities.

In this talk we show how to choose the weight function(s) so that the approximant reflects the discontinuities in the data. For doing so, we consider piecewise weight functions,of some order $\ell + 1$ of smoothness, that are themselves discontinuous. We take the weight functions as *Variable Scaled Discontinuous Kernels*, recently introduced in [2, 3], that enable us to reconstruct jump discontinuities. We will see that, this choice of weight functions provide tailored approximant that is useful to avoid overshoots near the edges of the underlying functions. Both theoretical and numerical analysis is provided.

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Software Implementation of the Partition of Unity Method

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The Partition of Unity (PU) scheme is a well-established and efficient kernel-based interpolation method. First introduced in the mid 1990s in [1], the PU method produces a global approximant by combining, via the use of compactly supported weights, many local fits [4]. Such a scheme is also rather popular for researchers working on local collocation schemes for PDEs. The PU method organizes the initial set of scattered data, that lay on a multivariate domain, into several patches. Then, for each of those patches it solves a small interpolation problem. A key step in its implementation is thus the one of efficiently distributing the scattered data into the different patches. A Matlab implementation of the PU scheme, based on the so-called kd-tree partitioning data structures (see [3]), already exists but it is not exploitable on recent Matlab releases. With this motivation, we propose an effective implementation of the PU scheme based on what we call the integer-based routines. The aim of this talk is to discuss the detailed implementation of the algorithm whose description was briefly treated in [2]. Moreover, motivated by the growing interest of the kernel community towards Python packages for machine learning, we also developed a Python implementation of the PU scheme. Finally, some experiments and comparisons between the two software implementations will be presented.

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Choosing the Shape Parameter Optimally in the RBF Collocation Method

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In this talk we totally discard the traditional trial-and-error algorithms of choosing acceptable shape parameter c in the multiquadrics

$$\phi(x) := -\sqrt{c^2 + \|x\|^2}$$

when dealing with differential equations. Instead, we predict the optimal value of c directly by the MN-curve theory and hence avoid the time-consuming steps of solving a linear system required by each trial of the c value in the traditional methods. The quality of the c value thus obtained is supported by the newly born rigorous choice theory of the shape parameter. Experiments show that the approximation error of the approximate solution to the differential equation is very close to the best approximation error among all possible choices of c.

The differential equation dealt with is the Poisson equation. A standard 3D Poisson equation is of the form

$$\begin{cases} u_{xx}(x,y,z) + u_{yy}(x,y,z) + u_{zz}(x,y,z) = f(x,y,z) & \text{for } (x,y,z) \in \Omega \backslash \partial\Omega, \\ u(x,y,z) = g(x,y,z) & \text{for } (x,y,z) \in \partial\Omega, \end{cases}$$

where Ω is the domain with boundary $\partial\Omega$, and f, g are given functions. A natural extension to d dimensions can be easily understood by replacing (x,y,z) with $(x_1, \cdots, x_d)$ and letting $\Omega \subseteq R^d$. The approximate solution will be of the form

$$\hat{u}(x) := \sum_{i=1}^{N} \lambda_i \phi(x - x_i) + p(x),\ x \in R^d,$$

where $\phi(x)$ is just the multiquadrics, $p(x)$ is a zero degree polynomial in R^d and hence a constant λ_0, and $X = \{x_1, \ldots, x_N\}$ is a set of points scattered in the domain without meshes. An unorthodox way even drops λ_0 and lets it be 0.The constants $\lambda_i,\ i = 0, \cdots, N$, are chosen so that $\hat{u}(x)$ satisfies the differential equation (including boundary conditions) at the points $x_i,\ i = 1, \cdots, N$, called the collocation points. The parameter c is chosen according to the MN curves which can be easily sketched by Matlab or Mathematica. If the minimum of the function MN(c) occurs at $c = c^*$, then c^* is just the optimal choice of c.

RBF interpolation and SSIM

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In the field of image reconstruction, the Structural Similarity index (SSIM) is widely used in assessing the similarity between two images. In this talk, first we present and discuss the extension of such index to the *continuous* framework, that is the continuous SSIM (cSSIM) [1]. Then, we relate this index to both the infinity and the L2-norm and, by focusing on the framework of RBF interpolation, we analyse the convergence rate in terms of the cSSIM, also by introducing a weighted version of such index. We provide a concrete explanation about the well-known fact that some images may be close in terms of the SSIM but not with respect to the L2-norm [2]. We show some numerical experiments that confirm the theoretical findings.

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A Novel Local Radial Basis Function Collocation Method for Multi-Dimensional Piezoelectric Problems

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The local radial basis function collocation method (LRBFCM), a strong-form formulation of the meshless numerical method, is proposed for solving piezoelectric medium problems. The proposed numerical algorithm is based on the local Kansa method using variable shape parameter. We introduce a novel technique for the determination of shape parameter in the LRBFCM, which leads to greater accuracy, and simplicity. The implemented algorithm is first verified with a 2D Poisson equation. Then, we employed LRBFCM in a numerical simulation for 2D and 3D piezoelectric problems involving mutual coupling of the electric field and elastodynamic equations for the mechanical field. The presented meshless method is verified using corresponding results obtained from the finite element method and moving least squares meshless local Petrov–Galerkin method [2]. In particular, the 2D piezoelectric problem is verified with an exact solution from [1].

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On the improvement of the triangular Shepard method by non conformal elements

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Most classical numerical methods for approximation of a multivariate function (or integrals of it) use function values at sample points. However, as shown in [2, 3], in many practical problems, the available data are not restricted by function evaluations, but contain several integrals over certain hyperplane sections, or, more generally, over smooth surfaces in $\mathbb{R}^d$. In such cases, generalizations of the existing theory and algorithms of approximation operators are required, which are based on the enriched set of data. In this work, we focus on this problem in the two dimensional case, in the setting of scattered data. More precisely we construct new Shepard type approximation operators, based on new enrichments of the standard linear triangular element, using polynomial functions. In line with previously considered improvements of the triangular Shepard method [4, 5], these enriched triangular elements will be blended by using triangular Shepard basis functions [1]. Numerical results are provided.

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Tchakaloff-like polyhedral quadrature with and without tetrahedralization

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The purpose of this talk is to show how to determine cubature rules on polyhedra with a certain algebraic degree of precision, positive weights and internal nodes, following two different approaches. In the first one we adopt a classical technique, based on tetrahedralization and the application of almost-minimal rules on each tetrahedron. In the second one, we show an alternative approach without tetrahedralization based on the divergence theorem, on a result of Wilhemsen [3] and on an indomain routine over polyhedra. As soon as these rules are available, we compute the nodes and weights of a low-cardinality positive quadrature formula by means Caratheodory-Tchakaloff cubature compression via NNLS (see, e.g. [1], [2]). Finally, we present several numerical tests, in order to assess the quality of our compressed formulas.

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On representation of derivatives using generalized Kantorovich sampling operators

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In this talk we consider representation of derivatives using generalized Kantorovich sampling operators, introduced in [1]. We consider also a construction of fractional derivative masks for image edge detection and enhancement, based on generalized Kan-torovich sampling operators. Edge detection and image enhancement methods, based on derivatives are well-known. Recently there is development to use several generalized fractional derivatives instead of the classical derivetaives of order 1 and 2 (see [2] and references cited there). Generalized sampling operators are a natural way to represent images. Such representation gives us a possibility to define different derivative masks. In fact, if the kernel is Hann kernel, we get the Sobel masks. Taking into account how well the generalized sampling operators allow to construct classical derivative masks, we use them also for fractional derivatives. We use generalized Kantorovich sampling operators, for more flexibility of the construction of masks.

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A fourth order scheme for nonlocal diffusive equations

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For some $a > 0$, in $x \in [-a, a]$, $t \geq 0$, we consider the diffusion equation with a nonlocal type kernel, where the function $\varphi(\cdot)$ is assumed to be sufficiently smooth:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} - u(x,y)J(u,x,t), \quad J(u,x,t) = \int_{-\infty}^{\infty} \varphi(y-x)u(y,t)\, dy \tag{3}$$

and with suitable initial and boundary conditions.

We propose a numerical method which is a combination of a "discrete-collocation" method in space and of the classical Runge-Kutta 4-5 method in time. We collocate the equation on a set P of selected equispaced points in $[-a, a]$. The second partial derivative with respect to space is discretized by suitable divided difference schemes of order 4. The integral J is approximated by means of a quadrature formula based on the generalized (iterated) Bernstein polynomial approximation, with the advantage of using the same set P of equispaced points in $[-a, a]$. The main feature of this formula lies in the high approximation order for smooth functions: more precisely, if the function to be integrated has $2r$ continuous derivatives, the convergence order of the quadrature rule is r. Matching this feature with the order chosen for the discretization of the derivatives, we obtain a system of ordinary differential equations in time. We finally proceed to integrate it by applying the Runge-Kutta scheme of order 4-5. Some numerical results will be shown.

S3. Functional Analysis, Approximation Theory and Differential Equations

The Session topics include, but are not limited to

- new iterative schemes to approximate fixed points of nonlinear mappings, common fixed points of nonlinear mappings or semigroups of nonlinear mappings
- iterative approximations of zeros of accretive-type operators
- iterative approximations of solutions of variational inequalities problems or split feasibility problems and applications
- optimization problems and their algorithmic approaches
- fixed point of nonlinear operators in cone metric spaces with applications and fixed points of nonlinear operators in ordered metric spaces with applications
- applications of Fixed Point Theory to establishing the existence of the solutions of differential equations or integral equations and inclusions
- other theoretical methods for differential and integral equations
- operator equations and inclusions in function spaces

Organizers:
Vittorio Colao, University of Calabria
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The New Aspect to Fixed Point Theory on Ultrametric Space

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The pioneer of fixed point theory in metric space is S. Banach [1] in which uniqueness of fixed point on complete metric spaces was proved for every contractive mapping. Banach's result has a crucial role in the theory in view of to present a way to find the fixed point of corresponding mapping as well as its existence and uniqueness. After the remarkable applications of fixed point theory in many branches, especially, integral equations, differential equations, numerical analysis, graph theory, etc., the theory has been extensivelly studied by researches for different contraction mappings in different type spaces.

Among the others, here, we consider one of these considerations presented by Gajic [2] on ultrametric space.The space, roughly speaking, is a special type of a metric space which is constructed by a stronger triangular inequality than classical ones. Considering the spherically completeness of a given ultrametric space, Gajic [2] obtained a fixed point theorem with uniqueness for the mappings.

In this talk, we aim to obtain some fixed point theorems using rational type F-contraction multivalued mapping on a spherically complete ultrametric space and give some corollaries and examples.

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BV spaces, sampling-type operators and approximation methods

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We will present some approximation results in the frame of the spaces of functions of bounded variation in multidimensional asset, by means of the variation in the sense of Tonelli. In particular, we obtain estimates and convergence in variation for some classes of sampling-type operators that have deep relations, from both theoretical and applicative aspects, with Approximation Theory, Signal and Image Processing. The natural geometrical aspects connected to the definition and construction of the Tonelli variation allow us to discuss also some applicative connections of the results to some problems of Digital Image Processing.

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How to measure the noncompactness of operators

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There are various methods to measure how "far" an operator in a normed space is from being compact. The most important tools are the essential norm (for linear operators) and the measure of noncompactness (for nonlinear operators).

We illustrate this by means of two linear operators, viz. the *multiplication operator*

$$M_\mu x(t) := \mu(t)x(t) \qquad (\mu : [0,1] \to \mathbb{R} \text{ given})$$

and the *substitution operator*

$$S_\varphi x(t) := x(\varphi(t)) \qquad (\varphi : [0,1] \to [0,1] \text{ given})$$

in the function spaces $C[0,1]$ with norm $\|x\|_C := \max\{|x(t)| : 0 \leq t \leq 1\}$ and $BV[0,1]$ with norm $\|x\|_{BV} := |x(0)| + var(x;[0,1])$. Combing this with analogous results for nonlinear superposition operators one may obtain existence results for boundary value problems.

This is joint work with S. Reinwand (Würzburg), L. Angeloni and G. Vinti (Perugia), and T. Dominguez Benavides (Sevilla).

Spectral analysis of a difference equation with interface conditions and hyperbolic eigenparameter on the whole axis

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Difference equations with interface conditions are a tool for mathematically explaining processes that are subject to sudden changes. These sudden changes depend on external factors and are negligibly short compared to the whole time. These equations were examined in detail by [5, 6, 7]. Recently, many researchers attach importance to such problems. Because difference equations with interface conditions have emerged in many areas of mathematical modeling such as physics, chemical, biotechnology, industrial robotics, ecology, population dynamics, optimal control, industrial robotics, medicine, control theory and so forth [1, 2, 3, 4, 8]. Although the theory of difference equations with interface conditions has many applications, there are insufficient studies examining the spectral analysis of these problems. This work investigates spectral analysis of a difference equation with interface (discontinuity) conditions and hyperbolic parameters on the whole axis. Firstly, we introduce the solutions of this difference equation. Then, we obtain resolvent operator, Green function and continuous spectrum by using these solutions. Finally, we present a condition which guaranties that the difference equation with interface condition has finite number of eigenvalues and spectral singularities with finite multiplicities.

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On Ulam stability of some linear difference equations

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An equation is called Ulam stable if for every approximate solution of it there exists an exact solution near it. We present some results on Ulam stability for some linear difference equations.

In a Banach space X the linear difference equation with constant coefficients

$$x_{n+p} = a_1 x_{n+p-1} + \ldots + a_p x_n,$$

is Ulam stable if and only if all the roots of the characteristic equation are not situated on the unit circle. In this talk, we present some stability results and the best Ulam constant for this equation.

Evolution equations with superlinear growth

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In this talk we present existence results [1] for mild solutions of the following partial differential equation of parabolic type

$$u_t = \Delta u + h(t, x, u(t, x)) \qquad \text{for} \qquad (t, x) \in (0, T) \times \Omega$$

coupled with Dirichlet boundary conditions on $\partial\Omega$ and a nonlocal initial condition $u(0, \cdot) = g(u)$ described by a map $g : C([0, T]; L^p(\Omega)) \to L^p(\Omega)$ with $2 \leq p < \infty$ where $\Omega \subset \mathbb{R}^k$ is a bounded domain with C^2-boundary. Here $h : [0, T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a given map with superlinear growth. The nonlocal condition includes as particular cases the Cauchy multipoint problem, the weighted mean value problem, and the periodic problem. Existence results are obtained by means of a Leray-Schauder continuation principle, transforming the above problem to an ordinary differential equation in the abstract setting given by the Banach space $L^p(\Omega)$. Handling superlinear growth in this context is particularly challenging since the Nemytskii operator associated to the Carathéodory function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ maps the space $L^p(\Omega)$ continuously on itself if and only if h is sublinear, as stated in Vainberg's theorem. We overcome this difficulty exploiting the compactness and the regularity properties of the semigroup generated by the Laplacian operator and constructing a suitable approximation technique. This approach was developed in [2] for the Cauchy problem and is extended here to the nonlocal one.

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Function spaces via heat kernel estimates

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I will discuss a theory of function spaces defined in terms of a self-adjoint operator whose heat kernel satisfies Gaussian estimates together with its derivatives. This includes inhomogeneous and homogeneous Besov and Triebel–Lizorkin spaces on Lie groups and Grushin settings [1, 2, 3].

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On Arzelà-Ascoli Theorem. Dualizing compactness property.

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We consider a metric space (U, ρ), two abitrary nonempty sets X, Y, a map (duality map)

$$D : X \times Y \to U$$

such that the partial functions D_x, D_y on Y, respectively X are totally bounded.

If d', d'' are the metrics on X, respectively Y given by:

$$d'(x', x'') = \sup\{\rho\left(D_{x'}(y), D_{x''}(y)\right); y \in Y\}$$

$$d''(y', y'') = \sup\{\rho\left(D_{y'}(x), D_{y''}(x)\right); x \in X\}$$

then the following assertions hold:

1. *The metric space (X, d') is totally bounded iff the metric space (Y, d'') is totally bounded.*

2. *The family $(D_x)_x$ of functions on Y has "finite small oscillations" iff the family $(D_y)_y$ has a similar property on X.*

The well-known assertions: Schauder theorem on compact linear operators, Scorohad compactness criterion with respect to a specific distance on trajectories, the famous Arzelà -Ascoli theorem...may be derived from the above assertions.

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A new look on Korovkin Theorem

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Giving a sequence $P = (P_n)_n$ of kernels on a measurable space (or just a semigroup $(P_t)_{t\in(0,\infty)}$) we are interested to describe the "semi-excessive" functions w.r. to P, i.e. measurable functions f such that $\lim_{n\to\infty} P_n(f) = f$ (or $\lim_{t\to 0} P_t(f) = f$).

We extend in this frame the famous Korovkin result on the uniform convergence of $P_n(f)$ to f on a class of measurable functions f, but beside that, we give a pointwise convergence result which may be a useful tool in Probabilistic Potential Theory as well Right Processes.

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Completeness theorems for systems of particular solutions of partial differential equations

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The results presented here fit in the problem of the completeness as proposed by Picone. This means to show that a certain system of particular solutions of a PDE is complete in a functional space on the boundary of a domain. This approach to the problem of polynomial approximation is more sophisticated than the usual one, which extends the classical Mergelyan Theorem. These results are related not only to a partial differential equation, but also to a particular boundary value problem. In particular we present necessary and sufficient conditions for the completeness of polynomial solutions of a partial differential equations of higher order in any number of variables related to the Dirichlet problem. We shall give also some recent results obtained for systems, where very little is known.

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Gibbons' conjectures for the Allen-Cahn equation.

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In this talk we deal with the Gibbons' conjecture that is also known as the weaker version of De Giorgi's one. Our aim is to prove the validity of this conjecture in the quasilinear case, based on a joint work with A. Farina, L. Montoro and B. Sciunzi.

Variational problems with nonconstant gradient constraints

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Aim of the talk is to study a nonlinear variational problem with gradient constraints and homogeneous Dirichlet boundary condition.

Variational problems with gradient constraints have been intensively studied a few decades ago and have seen many progresses also recently (see [2] for an overview on free boudary problems and applications related to the nonconstant gradient constrained problem). Indeed, an important example among them is the well-kwon elastic-plastic torsion problem. An interesting property of gradient constrained variational problems is the connection with double obstacle problems, even if this equivalence is not true in the general case (see [1], and counterexamples in [3, 4] in the case of nonconstant gradient constraint $|Du| \leq g(x)$).

In the talk we show that a nonlinear monotone variational inequality with convex gradient constraints is equivalent to a double obstacle problem.

The existence of Lagrange multipliers is also proved.

The proof is based on a new strong duality principle, that works in infinite dimensional settings.

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On a nonlocal boundary value problem for mixed type equation

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In this study, the mixed parabolic-hyperbolic type equation with the initial and integral boundary conditions are examined in the rectangular domain. In [1]-[4], the mixed type equations have many interesting applications in gas dynamics, electromagnetic movement of liquid and similar non-homogeneous processes. The integral boundary conditions show that physical process is not only at the point but also at the whole object. In this study, the existence, uniqueness and stability problems of the solution of the mixed type boundary problem are discussed. These problems are generally solved using the maximum principle or integral equations for mixed type boundary problems. However, the methods of spectral analysis are applied in this study.

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Approximation of fuzzy numbers by truncated Favard-Szasz-Mirakyan operators of max-product kind

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The nonlinear Favard-Szasz-Mirakjan operators of max-product kind is introduced in [1]. In [2], the authors showed that the order of approximation by the truncated max-product Favard-Szász-Mirakjan operator is less than $C\omega(f, 1/n)$, (C=6). The aim of this note is to study the approximation of fuzzy numbers by truncated Favard-Szasz-Mirakyan operators of max-product kind.

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Integral representations for solutions of some BVPs for steady elastic oscillations

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We consider some basic three-dimensional boundary value problems (BVPs) for steady elastic oscillations. In particular we deal with the Dirichlet problem of representability of the solutions by means of a simple layer potential. The main result concerns the solvability of the boundary integral system of equations of the first kind; this is obtained by using the theories of differential forms and reducible operators. We also consider the traction problem; representability of its solution by means of a double layer potential is presented instead of the more usual simple layer potential. This talk is based on joint work with A. Cialdea and V. Leonessa ([1] and [2]).

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Regularity and Stability for a Convex Feasibility Problem

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The 2-sets convex feasibility problem aims at finding a point in the nonempty intersection of two closed convex sets A and B in a Hilbert space H. The method of alternating projections is the simplest iterative procedure for finding a solution and it goes back to von Neumann. Originally, he proved that the method of alternating projection converges when A and B are closed subspace. Then, for two generic convex sets, the weak convergence of the alternating projection was proved by Bregman in 1965 ([2]). Nevertheless, the problem of whether the alternating projections algorithm converges in norm for each couple of convex sets remained open till the counterexample given by Hundal in 2004. This example shows that the alternating projections do not always converge in norm. One of the most important sufficient condition ensuring the norm convergence of the alternating projections algorithm is based on the so called "regularity" property for the couple (A, B) ([1]). In this talk we consider two sequences of closed convex sets $\{A_n\}$ and $\{B_n\}$, each of them converging, with respect to the Attouch-Wets variational convergence, respectively, to A and B. Given a starting point a_0, we consider the sequences of points obtained by projecting on the "perturbed" sets, i.e., the sequences $\{a_n\}$ and $\{b_n\}$ defined inductively by $b_n = P_{B_n}(a_{n-1})$ and $a_n = P_{A_n}(b_n)$. i.e., for each $\{a_n\}$ and $\{b_n\}$ as above we have $\mathrm{dist}(a_n, A \cap B) \to 0$ and $\mathrm{dist}(b_n, A \cap B) \to 0$. We show that the regularity of the couple (A, B) implies not only the norm convergence of the alternating projections sequences for the couple (A, B), but also that the couple (A, B) is d-stable.i.e., for each $\{a_n\}$ and $\{b_n\}$ as above we have $\mathrm{dist}(a_n, A \cap B) \to 0$ and $\mathrm{dist}(b_n, A \cap B) \to 0$. Similar results are obtained also in the case $A \cap B = \emptyset$, considering the set of best approximation pairs instead of $A \cap B$. Finally, under appropriate geometrical and topological assumptions on the intersection of the limit sets, we ensure that the sequences $\{a_n\}$ and $\{b_n\}$ converge in norm to a point in the intersection of A and B. The talk is based on [4] and [3].

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Singular solutions to p-Laplacian systems.

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We consider positive singular solutions (i.e. with a non-removable singularity) of a system of PDEs driven by p-Laplacian operators. We prove some fine regularity properties of the solutions, and then we show symmetry and monotonicity properties.

Applications of fibre contraction principle to some classes of equations

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By a new variant of fibre contraction principle ([4]) we give existence, uniqueness and convergence of successive approximations results for some functional equations. In the case of ordered Banach space, Gronwall-type and comparison-type results are also given.

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On the best constant in Ulam stability

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A linear operator is Ulam stable if for every approximate solution of the associated equation there exists an element in the kernel of the operator close to the approximate solution.

The linear differential operator with constant coefficients

$$D(y) = y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y, \quad y \in \mathcal{C}^n(\mathbb{R}, X)$$

acting in a Banach space X is Ulam stable if and only if its characteristic equation has no roots on the imaginary axis. We prove that if the characteristic equation of D has distinct roots r_k satisfying $\Re r_k > 0$, $1 \leq k \leq n$, then the best Ulam constant of D is

$$K_D = \frac{1}{|V|} \int_0^{\infty} \Big| \sum_{k=1}^{n} (-1)^k V_k e^{-r_k x} \Big| dx,$$

where $V = V(r_1, r_2, \ldots, r_n)$ and $V_k = V(r_1, \ldots, r_{k-1}, r_{k+1}, \ldots, r_n)$, $1 \leq k \leq n$, are Vandermonde determinants.

Explicit algebraic solutions, for low-degree polynomials, to Zolotarev's First Problem of 1868

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We report on our recent progress concerning the explicit algebraic solution of Zolotarev's First Problem (ZFP) of 1868 ([1], [2]), thus avoiding the application of elliptic functions. ZFP asks to determine, for $n \geq 4$ and $s > \tan^2\left(\frac{\pi}{2n}\right)$, the proper Zolotarev polynomial $Z_{n,s}$ which deviates least from zero in the uniform norm on $[-1, 1]$ among all polynomials of form $x^n + (-ns)x^{n-1} + \ldots$. By parametrization of algebraic curves, we have obtained in [4] a radical parametrization for $Z_{7,s}$ (currently the highest degree attacked by that method). Out of it, the solution of ZFP, for $n = 7$, can be recovered. In [3] we considered two alternative algebraic algorithms for explicitly solving ZFP (one was inspired by [5]). We now add a third one (inspired by [6]). Each algorithm creates a particular tentative form (depending on parameters α and β with $1 < \alpha < \beta$) of $Z_{n,s}$. To get the final form of $Z_{n,s}$ for the concretely chosen $n = n_0$ and $s = s_0$, the algorithms require as input compatible points $\alpha = \alpha_0$ and $\beta = \beta_0$ (depending on n_0 and s_0). In [3] we considered one variant how to determine α_0 and β_0. We now add two more variants. The variants involve Malyshev polynomials $F_n(\alpha)$ and $G_n(\beta)$, determinants with variable elements $d_{i,j}(\alpha, \beta)$, reduced relation curves $H_n(\alpha, \beta) = 0$, and function equations of form $s_n(\alpha, \beta) = s$. We show how to generate these terms by means of *Mathematica*-functions, e.g., GroebnerBasis. Pre-computed data to facilitate the computation of $Z_{n,s}$ and concrete examples, if $n \leq 13$, are provided and further existing non-elliptic approaches to ZFP are referenced.

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S4. Multivariate Polynomials in Approximation and Signal Analysis

The main goal of the session is to provide a platform for discussion of new developments and recent advances in the areas of approximation and interpolation of functions of several variables, extremal properties of multivariate polynomials and signal analysis.

Organizers:

Frank Filbir, Technische Universität München
András Kroó, Hungarian Academy of Sciences
Woula Themistoclakis, CNR, IAC

L_p Markov exponent of certain UPC sets

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We say that a compact set $\emptyset \neq E \subset \mathbb{R}^m$ satisfies L_p Markov type inequality (or: is a L_p Markov set) if there exist $\kappa, C > 0$ such that, for each polynomial $P \in \mathcal{P}(\mathbb{R}^m)$ and each $\alpha \in \mathbb{N}_0^m$,

$$\|D^\alpha P\|_{L_p(E)} \leq (C(\deg P)^\kappa)^{|\alpha|} \|P\|_{L_p(E)}, \tag{4}$$

where $D^\alpha P = \frac{\partial^{|\alpha|} P}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_m$.

Clearly, by iteration, it is enough to consider in the above definition multi-indices α with $|\alpha| = 1$. The inequality (4) is a generalization of the classical Markov inequality:

$$\|P'\|_{C([-1,1])} \leq (\deg P)^2 \|P\|_{C([-1,1])}.$$

In this talk we shall consider the following problem:

For a given L_p Markov set E determine $\mu_p(E) := \inf\{\kappa : E$ satisfies (4)$\}$.

Our goal is to establish L_p Markov exponent of the following domains

$$K := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ ax^k \leq y \leq f(x)\},$$

where $k \in \mathbb{N}$, $k \geq 2$, $a > 0$ and $f : [0, 1] \to [0, \infty)$ is a convex function such that $f(1) > a$, $f'(0) = f(0) = 0$, $f'(1) < \infty$, and $(f)^{1/k}$ is a concave function on the interval $(0, 1)$.

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Łojasiewicz exponent and pluricomplex Green's function on algebraic sets

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The talk is based on the paper [1].

The classical invariance theorem for pluricomplex Green functions in $\mathbb{C}^N$ states that

If k, ℓ are positive integers and $f : \mathbb{C}^N \to \mathbb{C}^N$ is a holomorphic mapping, then the following conditions are equivalent:

(i) f is a polynomial mapping of degree at most ℓ and $\liminf_{\|z\|\to\infty} \frac{f(z)}{\|z\|^k} > 0$,

(ii) f is a proper mapping and for every compact set $K \subset \mathbb{C}^N$

$$k\, V_{f^{-1}(K)} \;\le\; V_K \circ f \;\le\; \ell\, V_{f^{-1}(K)} \qquad \textit{in} \;\; \mathbb{C}^N,$$

see [2], [3] or [4, Th.5.3.1]. The main objective of the talk is to present a generalization of this result to pluricomplex Green functions on algebraic sets.

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Efficient time-frequency representations of nonstationary signals for instantaneous frequencies estimation

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Instantaneous frequency (IF) estimation of non-stationary signals is required in many applications, such as radar and micro doppler systems, seismology, audio and speech processing, biology, biomedicine, air traffic control. To this aim sparse time-frequency representations that can deal with signals having non-separable components are necessary [1, 2, 5, 6].

In this talk, some recent results concerning the retrieval of good grid points for IF estimation are presented [3, 4]. They take advantage of a signal spectrogram time-frequency evolution law and reassignment procedures. Open issues are also discussed.

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An algorithm for constrained regression by penalized splines

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Regression splines are confirmed powerful and versatile to investigate the data structures, and predict data behaviour. Different penalized models are available in literature, so-called since balancing the regression fidelity and a weighted penalizing term. The most commonly used are P-splines [5] that are based on two main ingredients: polynomial B-splines and a second order discrete difference penalty. More recently [2] a new class of penalized splines, said hyperbolic-polynomial splines (HP-splines), combines the advantages of P-splines with the idea of a data-driven selection of space parameters, making them more suitable for capturing exponential trends. Actually, P-splines can be recognized as one of the most successfully smoothers used in a wide range of applications. For example, in social and behavioural sciences, constrained P-splines are definite based on theoretical hypotheses regarding their shape and monotonicity, translated into local and global constraints on the successive derivatives of the functional model [1]. Moreover, P-splines are used for Bayesian spectral density estimation; e.g. in [6] the authors propose statistic techniques for a data-driven selection of the spline knots placement, so giving up the uniform distribution of the knots which guarantees analytical reproduction properties (assured both for P-splines [4] and for HP splines [3]). We formulate a constraint optimization problem to make a penalized spline positive, so suitable for density estimation, and propose a greedy-type algorithm to dynamically tune the model shape, while preserving positiveness and space reproduction properties. A theoretical result concerning the bounded variation of the P-spline model is also given and drives the selection.

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Bernstein polynomials and subdivision schemes for the reconstruction of binary images

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We propose a new method for the stable reconstruction of a class of binary images (modelled as characteristic functions of algebraic domains) from a small number of measurements. Algebraic polynomials and the corresponding image moments are represented in terms of bivariate Bernstein polynomials. We illustrate a strategy for the computation of the coefficients involved in such a representation by means of refinable function kernels associated to convergent polynomial-generating subdivision schemes. The computational procedure relies on the construction of a quasi-interpolation operator whose coefficients are the solution to a linear system. Our approach is robust to noise, computationally fast and simple to implement. The performance of the reconstruction algorithm from noisy samples is illustrated through the results of extensive numerical experiments.

This is a joint work with Demetrio Labate and Wilfredo Molina (University of Houston, Texas, USA).

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Multivariate fakes nodes approach

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The concept of mapped bases has been widely studied, but all the proposed methods show convergence provided that the function is resampled at the mapped nodes. In applications, this is often physically unfeasible. We discuss the extention of the so-called *mapped bases without resampling* interpolation, also known as *Fake Nodes Approach* (FNA) [1, 2], to *any* basis and dimension. The univariate case has been discussed in [1] and some of its applications have been collected in the recent paper [2].

It is a common practice in multimodal medical imaging to undersample the anatomically-derived segmentation images to measure the mean activity of a co-acquired functional image. The applicarion to medical image resampling is then presented, showing that the FNA is an effective way to reduce the Gibbs effect when oversampling the functional image [4].

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Marcinkiewicz-Zygmund inequalities for scattered and random data on the d-dimensional unit sphere

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The recovery of functions and estimating their integrals from finitely many samples is one of the central tasks in approximation theory. MarcinkiewiczZygmund inequalities provide answers to both the recovery and the quadrature aspect. In this paper, we focus on the q-dimensional sphere $\mathbb{S}^q$, and investigate how well continuous L^p-norms of polynomials f of maximum degree n on the sphere $\mathbb{S}^q$ can be discretized by positively weighted L^p-sum of finitely many samples, and discuss the relationship between the offset between the continuous and discrete quantities, the number and distribution of the (deterministic or randomly chosen) sample points $x_1, \ldots, x_N$ on $\mathbb{S}^q$, the dimension q, and the polynomial degree n.

Heat-diffusion semigroup and other translations

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We introduce general translations as solutions to Cauchy or Dirichlet problems. This point of view allows us to handle the heat-diffusion semigroup as a translation. With the given examples Kolmogorov-Riesz characterization of compact sets in certain L^p_μ spaces are given. Pego-type characterizations are also derived. Finally for some examples the equivalence of the corresponding modulus of smoothness and K-functional is pointed out.

Polynomial meshes on algebraic hypersurfaces

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Polynomial meshes (called sometimes 'norming sets') are nearly optimal for uniform least squares approximation and contain interpolation sets nearly as good as Fekete points of the domain. They play a relevant role in recent multivariate interpolation and approximation. Optimal meshes have been constructed by different analytical and geometrical techniques on many polynomially determining compact sets. Regarding subsets of algebraic varieties polynomial meshes are known only for a few compacts like sections of a sphere, a torus, a circle and curves in $\mathbb{C}$ with analytic parametrization. In these cases, polynomial meshes are transferred by some analytical map from certain polynomially determining set with sufficiently relevant meshes. We give a general construction of polynomial weakly admissible meshes on compact subsets of arbitrary algebraic hypersurfaces in $\mathbb{C}^{N+1}$. They are preimages by a projection of meshes on compacts in $\mathbb{C}^N$. These meshes are optimal in some cases. We present also partial results for algebraic sets of codimension greater than one. We give some examples of optimal polynomial meshes and weakly admissible meshes on compact subsets of algebraic sets.

On Bernstein- and Marcinkiewicz-type inequalities on multivariate C^α-domains

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We prove new Bernstein and Markov type inequalities in $L^p, 1 \leq p < \infty$ spaces associated with the normal and the tangential derivatives on the boundary of a general compact C^α-domain with $1 \leq \alpha \leq 2$. These estimates are also applied to establish Marcinkiewicz type inequalities for discretization of L^p norms of algebraic polynomials on C^α-domains with asymptotically optimal number of function samples used. This extends L^p tangential Bernstein type and Marcinkiewicz type inequalities given in [1] on a general compact C^2 domain. In case when $p = \infty$ similar Bernstein type inequalities and asymptotically optimal discretization meshes on C^α-domains were given earlier in [2].

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Numerical methods for 2D linear Fredholm integral equations on curvilinear polygons

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In this talk, we consider the numerical approximation of 2D Fredholm integral equations of the second kind, defined on a curvilinear polygon S (general bi-dimensional domain whose boundary is a piecewise smooth Jordan curve),

$$f(x,y) - \mu \int_S k(x,y,s,t) f(s,t)\, dsdt = g(x,y), \qquad (x,y) \in S,$$

where g and k are given functions defined on S and S^2, respectively, μ is a fixed real parameter and f is the unknown function in S.

The literature about 2D Fredholm integral equations is not very wide. The methods avaiable mainly consider the case of rectangular domains (see for instance [1] and the references therein). If a global approximation strategy is chosen, the most simple and powerful approach seems to be Nyström methods based on cubature rules obtained as the tensor product of two univariate rules.

On the contrary no global approach was proposed for the case of curvilinear domains that cannot be transformed in a square. On the other hand even if the transformation is possible, the smoothness of the known functions could be not preserved and this can produce a severe loss in the rate of convergence of the method.

Here we propose a numerical method of Nyström type based on cubature formulas introduced in [2]. This choice allows to treat general curvilinear domains, to use a global approximation approach and avoids the loss of convergence due to transformations.

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Direct method for function approximation on data defined manifolds, II

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In theoretical analysis of function approximation in the context of machine learning, one of the standard assumptions in order to avoid the curse of dimensionality is the manifold assumption; i.e., one assumes that the data is sampled from an unknown sub-manifold of a high dimensional Euclidean space. A great deal of research deals with obtaining information about this manifold, such as the eigen-decomposition of the Laplace-Beltrami operator or coordinate charts. The theory of function approximation based on this preliminary information is also well studied. Since the manifold is unknown, this two step approach implies some extra errors in the approximation stemming from the approximation of the basic quantities from the data in addition to the errors inherent in function approximation. In [1], HNM has proposed a one-shot direct method to achieve function approximation without knowing anything about the manifold other than its dimension. However, one cannot pin down the class of approximants used in that paper.

In this paper, we view the unknown manifold as a sub-manifold of an ambient hypersphere and study the question of constructing a one-shot approximation using the spherical polynomials based on the hypersphere; again, our approach does not require pre-processing of the data to obtain information about the manifold other than its dimension. We give optimal rates of approximation for relatively "rough" functions. The class of approximants is the restriction of the spherical polynomials to the unknown manifold, but we need not use this fact in our construction.

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Prony-Type Polynomials

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Let us consider N-sparse bivariate exponential sum,

$$f(\mathbf{k}) = \sum_{j=1}^{N} a_j \exp(-\mathrm{i}\langle \boldsymbol{\omega}_{\mathrm{j}}, \mathbf{k}\rangle) + \epsilon(\mathbf{k}),$$

where $a_1, \ldots, a_N \in \mathbb{C} \setminus \{0\}$, $\mathbf{k} \in \mathbb{Z}^2$, $\langle \boldsymbol{\omega}_j, \mathbf{k}\rangle$ denotes the inner product of $\mathbf{k}$ and $\boldsymbol{\omega}_j$ and $\epsilon(\mathbf{k})$ is are a random variable. The problem of parameter estimation of the exponential sum is to determine approximately elements $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_N \in (0, 2\pi]^2$ out of finitely many noisy samples of f.

Inspired by the one-dimensional approach developed in [1], we propose to use the method of Prony-type polynomials, when the parameters $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_N$ can be recovered as a set of common zeros of Prony-type polynomials, some bivariate polynomials of an appropriate multi-degree. Numerical experiments show the PTP method is more stable in the presence of noise than other methods. Moreover, using an autocorrelation sequence in the PTP approach allows us to improve significantly the stability of the method in the noisy data case.

This is joint work with Jürgen Prestin (Institute of Mathematics, University of Lübech).

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S5. Numerical Advances in Differential Equations

The main aim of this session is to bring together researchers in the field of numerical methods for differential equations to present their recent results and exchange ideas for new developments in theory and related applications, with particular emphasis in areas such as ordinary and stochastic differential equations.

Organizers:
Lidia Aceto, University of Piemonte Orientale
Carmela Scalone, University of L'Aquila
Zdzislaw Jackiewicz, Arizona State University

New Developments in the Numerical Solution of Sturm-Liouville Problems by High Order Finite Difference Schemes

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The matlab code HOFiD (based on high order finite difference schemes) has been successfully used to solve several kind of Sturm-Liouville and Multiparameter Spectral problems [1, 2].

Starting from the research in [3], we now propose an update in order this code also solves problems with trapezoidal or piecewise continuous potentials as well as eigenparameter-dependent boundary conditions.

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Convergence of the piecewise orthogonal collocation for computing periodic solutions of delay equations

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Realistic models from several areas of biology, such as structured population dynamics or epidemiology, are often based on delay equations. Due to the infinite dimension of the relevant dynamical systems, the dynamics of such models can generally not be studied analytically and must then be approximated numerically. An important target in this context is represented by the computation of periodic solutions. As for Retarded Functional Differential Equations (RFDEs), the classical piecewise orthogonal collocation proposed in [4] for computing periodic solutions has only recently been proven to be convergent in [2], following the abstract approach in [5] for general boundary value problems defined by delay equations. The method can also be extended to Renewal Equations (REs), and its convergence is proved in [1, 3]. We take a further step by describing the extension of the method to coupled systems of RFDEs and REs. In this talk, I present the main ideas behind the proof of its convergence and emphasize the challenges that emerge with respect to the cases of RFDEs and REs separately. Finally, I show some numerical experiments that further validate the theoretical results.

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Adaptive Exponential Splitting

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For the efficient numerical integration of evolutionary differential equations with additive right-hand side,

$$\partial_t u = F_1(t, u) + F_2(t, u) + \ldots,$$

approximation by exponential splitting methods based on the solution and systematic recombination of sub-problems is often a convenient choice. For the purpose of reliable adaptive stepsize control, several techniques for practical local error estimation are introduced and discussed in [1], and further works are devoted to particular applications, e.g., Schrödinger type problems or hyperbolic problems [2]. The employed techniques comprise optimized pairs of schemes, Milne-type estimators, and defect-based approaches, and are used with schemes of various approximation orders. Also, more than two suboperators are treated, see [2, 3] for the case of three sub-operators F_j. Coefficients of various optimized methods are collected at [4].

In the simulation of magnetohydrodynamics, it is natural to split the vector field into 4–8 operators, even. Additionally, certain positivity conditions on the coefficients need to be satisfied. We discuss the construction of methods and present numerical results for hyperbolic test problems.

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Asymptotic preserving and asymptotic accurate schemes for hyperbolic systems with stiff hyperbolic or parabolic relaxation

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Several systems of evolutionary partial differential equations may contain stiff terms, which require an implicit treatment. Typical examples are hyperbolic systems with stiff hyperbolic or parabolic relaxation characterized by a relaxation parameter ε. In the hyperbolic-to-hyperbolic relaxation (HSHR) a natural treatment consists in adopting implicit-explicit (IMEX) schemes, in which the relaxation is treated by an implicit scheme, while the hyperbolic part is treated explicitly [1]. In the hyperbolic-to-parabolic relaxation (HSPR) standard IMEX methods relax to an explicit scheme for the parabolic limit, thus suffering from parabolic CFL restriction. In [2, 3] this drawback has been overcame by a penalization method, consisting in adding and subtracting the same term, so that the system appears as the limit relaxed system plus a small perturbation. Furthermore, in [4] a unified IMEX approach has been introduced for systems which may admit both limits. This latter approach generalizes the two ones: HSHR and HSPR. All these approaches are capable to capture the correct asymptotic limit of the system when $\varepsilon \to 0$, i.e., the scheme is *asymptotic preserving* (AP) independently of the scaling used. However, the AP property guarantees only the *consistency* of the scheme in the stiff limit $\varepsilon \to 0$, but it does not not imply in general that the scheme preserves the order of accuracy in time in the limit and the order of accuracy may drop to low orders. In the literature of hyperbolic system with stiff relaxation this *order reduction phenomenon* is extensively studied and cured, see for example [1]. A scheme that preserves the order of accuracy in time in the limit is said asymptotically accurate (AA). In this talk we show that under several assumptions on the IMEX scheme, all the numerical approaches used to solve hyperbolic systems with stiff relaxation are both AP and AA, i.e., they maintain the correct order of accuracy of the original IMEX scheme in the limit of the relaxation parameter ε.

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Search for implicit-explicit general linear methods with inherent Runge-Kutta stability

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Many practical problems in science and engineering are modeled by large systems of ordinary differential equations (ODEs) which arise from discretization in space of partial differential equations (PDEs) by finite difference methods, finite elements or finite volume methods, or pseudospectral methods. For such systems there are often natural splittings of the right hand sides of the differential systems into two parts, one of which is non-stiff or mildly stiff, and suitable for explicit time integration, and the other part is stiff, and suitable for implicit time integration. The efficient solution can be provided by implicit-explicit (IMEX) schemes.

In present research we consider the class of general linear methods (GLMs) for ordinary differential equations. We construct IMEX GLMs of order $p = 1, 2, \ldots, 4$ with desired stability properties. We assume that the explicit and implicit parts of the IMEX scheme have the same abscissa vector $\mathbf{c}$ and coefficient matrices $\mathbf{U}$ and $\mathbf{B}$. We look for implicit methods of order p and stage order $q = p$, which have the property of so-called inherent Runge–Kutta stability (IRKS), and which are A-stable and if possible, also L-stable. We also require that the vector of external approximation is of Nordsieck form. Next, we attempt to maximize the combined region of absolute stability. Finally, we apply constructed methods to a series of test problems.

This is a joint work with A. Cardone, Z. Jackiewicz and P. Pierzchała.

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Numerical computation of the basic reproduction number

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The twentieth century has witnessed the emergence of the basic reproduction number as a key player in assessing the growth of a population or the spread of a disease. Only in the nineties this quantity has been rigorously characterized as the spectral radius of a positive linear operator, promoting since then the use of increasingly realistic, yet more complicated, models. In this talk we would like to present some recent developments [1, 2, 3] in the numerical approximation of this number, first illustrating a spectrally accurate discretization framework and then discussing its convergence. As an application we consider models of epidemics structured by individual traits as, e.g., age or immunity.

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Multivalue numerical methods for stiff differential problems

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This talk focuses on the numerical solution of differential equations characterized by stiffness deriving from real phenomena and physical processes. Stiff problems can arise when dealing, for example, with partial differential equations with advection or diffusion terms. We show the application of numerical techniques leading to methods equipped with excellent stability properties, also capable of preserving the qualitative features of the solution [3, 4].

Recent numerical techniques for the construction of efficient and stable methods involve the modification of the classical coefficients which become Jacobian-dependent matrices [5]. In this field, the Time-Accurate and Highly-Stable-Explicit (TASE) Runge-Kutta methods are derived by modifying the problem to be solved by introducing an appropriate operator [2]. We show that it is possible to derive highly stable multivalue methods using the TASE technique and other methodologies [1, 4].

The presented results have been obtained in collaboration with Beatrice Paternoster, Leila Moradi and Giovanni Pagano (University of Salerno), Raffaele D'Ambrosio (University of L'Aquila), Fakhrodin Mohamadi (University of Hormozgan, Iran), Ali Abdi (University of Tabriz, Iran).

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Numerical conservations features of stochastic Hamiltonian systems

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We address our attention to the numerical approximation of stochastic Hamiltonian systems, both of Itô and Stratonovich types. It has been shown that, in the Itô case, a trace equation is satisfied, describing the linear growth of the expected Hamiltonian in time [1], while energy conservation is visible in the Stratonovich setting [5]. The first part of this talk is devoted to the analysis of the effectiveness of Monte Carlo estimates employed in the application of drift-preserving numerical schemes for Itô Hamiltonian systems [3], i.e., methods able to reproduce the trace equation along the numerical dynamics [1, 2]. In the second part of this talk, we aim to provide a characterization of the long-term behavior of numerical discretizations to such stochastic Hamiltonian systems [4] by means of the so-called *weak backward error analysis*. The key ingredient is the construction of weak stochastic modified equations associated to such numerical methods [4, 6, 7]. Finally, numerical experiments are also provided to confirm the theoretical results.

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A numerical procedure for fractional-time-space differential equations with the spectral fractional Laplacian

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We present a computationally effective procedure for numerically solving fractional-time-space differential equations with the spectral fractional Laplacian [1]. A truncated spectral representation of the solution in terms of the eigenfunctions of the usual integer-order Laplacian is considered. Time-dependent coefficients in this representation, which are solutions to some linear fractional differential equations, are evaluated by means of a generalized exponential time-differencing method [2], which presents some advantages in terms of accuracy and computational effectiveness. Rigorous a-priori error estimates are derived, and they are verified by means of some numerical experiments.

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Synchronous periodic orbits of PWS networks: theoretical and numerical aspects for asymptotic stability.

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We consider synchronous solutions of networks of piecewise smooth oscillators. The asymptotic stability of synchronous solutions is highly desirable but it is still an open problem how to ascertain it for networks of piecewise smooth oscillators. Two main difficulties must be overcome: i) The fundamental matrix solution is not unique in general; ii) The large dimension of the problem requires efficient numerical techniques. In this talk we address both issues and we extend the Master Stability Function algorithm of Pecora and Carroll to networks of piecewise smooth oscillators.

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Perturbative analysis of numerical discretization to stochastic Hamiltonian problems

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This talk will highlight recent results based on the study of numerical dynamics associated to discretization of stochastic Hamiltonian problems, since they are excellent models useful in a large number of applications, when the dynamics is subject to random perturbations. In particular, stochastic Hamiltonian problems are the most suitable candidates to conciliate classical Hamiltonian mechanics with the non-differentiable Wiener process, which describes the continuous innovative character of stochastic diffusion.
Our analysis is focused on the study of stochastic Runge-Kutta methods developed by Burrage and Burrage, obtained as a stochastic perturbation of classical symplectic Runge-Kutta methods. In particular, we are interested in understanding whether these methods are capable to maintaining the linear drift visible in the expected value of the Hamiltonian. The analysis shows Runge-Kutta methods present an error that increases with the parameter ϵ, being ϵ the amplitude of the diffusive part of the problem. Through a perturbative theory, we investigate the reason of this behaviour, due to the presence of a secular term $\epsilon\sqrt{t}$ that destroying the overall conservation accuracy. Numerical tests confirm the theoretical analysis.

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PDE-convergence of AMF-W methods for parabolic problems

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The time integration of space-discretised parabolic problems (in m spatial dimensions) on rectangular-like domains subject to Dirichlet boundary conditions is considered. The time integration is carried out by using s-stage AMF-W-methods, which are ADI (alternating direction implicit) type integrators. Optimal results of PDE-convergence (convergence in time independently of the spatial resolution) in the Euclidean norm for the case of $m = 2$ are given [1, 2]. Most of this results can be extended to the case $m > 2$ [3]. Some numerical experiments on linear problems confirm the theory.

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Generalized TASE-RK methods for stiff problems with application to the numerical solution of parabolic PDEs.

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The so-called family of Time-Accurate and Stable Explicit (TASE) methods for the numerical integration of Initial Value Problems in stiff Ordinary Differential Equations (ODEs) was recently introduced in [1]. Such methods consider a base explicit Runge-Kutta (RK) method whose stability properties are improved by multiplying the vector field of the underlying ODE by a certain operator which approximates the identity mapping up to a given order p. This family of methods was further extended to a wider class of TASE operators in [2], where some classical linear stability properties were studied, improving the stabilization of some classical explicit Runge-Kutta up to order four. The TASE operators considered for a given explicit method of order p and p stages ($1 \leq p \leq 4$) require the solution of p linear systems for each internal stage.

In this talk, generalized TASE-RK methods are considered in order to improve the efficiency of the TASE approach while retaining the order of consistency and good linear stability properties. Since these methods are linearly implicit, connections to the class of W-methods [3] are established. Furthermore, an extension of the TASE approach with the Approximate Matrix Factorization technique is proposed in order to deal with the numerical solution of large ODEs coming from the spatial discretization of parabolic Partial Differential Equations with time-dependent boundary conditions in several spatial dimensions.

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A new framework for polynomial approximation to differential equations

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We discuss a new framework for the polynomial approximation to the solution of initial value problems for ordinary differential equations (ODEs)

$$\dot{y}(t) = f(t, y(t)), \qquad t \in [t_0, T], \qquad y(t_0) = y_0 \in \mathbb{R}^m, \tag{5}$$

and delay differential equations (DDEs) in the form,

$$\begin{array}{rcll} \dot{y}(t) & = & f(t, y(t), y(t-\tau)), & t \in [t_0, T], \\ y(t) & = & \phi(t), & t \in [t_0 - \tau, t_0], \end{array} \tag{6}$$

The framework is based on a truncated expansion of the vector field along an orthonormal basis, which projects the differential problem onto a finite dimensional vector space. This procedure leads to a new class of numerical methods that may be regarded as a perturbation of the original differential problem. Consequently, a perturbation analysis has been successfully employed to understand how the solutions of the two problems are related.

The approach has been initially devised for problem (5) ([1, 2, 3]), and very recently extended to delay differential equations in the form (6) ([5, 4]). Relevant classes of Runge-Kutta methods can be derived within this framework.

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Self Starting General Linear Methods for Ordinary Differential Equations

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We have recently focused our attention on using general linear methods (GLMs) as a framework to analyze and generalize existing classes of numerical methods for ordinary differential equations. In this work we present the class of Self Starting GLMs, whose name point out one of their main features. Indeed, although they are multi-stage multi-step methods, they do not require any additional starting procedure. In particular, after presenting the general formulation, we focus on a subclass with a structure that is very similar to Runge-Kutta methods. With this approach, we show how some properties of these last methods can be improved, keeping similar computational costs. This analysis indicates that the proposed methods may have better accuracy and stability properties, such as, for example, larger stability regions in the case of explicit methods, or stage order greater than one for singly diagonally implicit methods.

The possibility of identifying good families of methods with a larger number of degrees of freedom can also have implications in the field of time discretization of partial differential equations. For example, Self Starting GLMs allow the determination of new efficient and highly stable Implicit-Explicit and Strong Stability Preserving methods.

Finally, we report numerical experiments which confirm that Self Starting GLMs are competitive with Runge-Kutta methods and can have better performance on nonstiff, mildly stiff and stiff problems.

Investigating the stability of periodic neutral renewal equations via Floquet multipliers

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The main classes of retarded functional equations are delay differential equations, which describe the solution by prescribing the values of its derivative, and renewal equations, which directly prescribe the values of the solution itself. Renewal equations proper are typically formulated as integral equations; neutral renewal equations (NRE), instead, involve also values of the solution at discrete times in the past. Even though NRE are quite important in biological modeling, until the recent work [1] little was known about their dynamics.

In this work we investigate the stability of autonomous and periodic NRE of the type $x(t) = f(t)x(t - \tau)$ from a theoretical point of view, performing numerical experiments both to exemplify our findings and to guide our investigation.

Our numerical approach is based on the pseudospectral collocation technique of [2, 3] for discretizing monodromy operators and computing Floquet multipliers, although a proof of Floquet theory for NRE is currently lacking.

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A minimum-time obstacle-avoidance path planning algorithm for unmanned aerial vehicles

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We present a new method to determine an unmanned aerial vehicle (UAV) trajectory that minimizes its flight time in the presence of avoidance areas and obstacles. The optimal control problem is numerically solved using the indirect method with a set of penalty functions embedded in a suitable continuation technique [1, 2]. The arising nonlinear boundary value problems are efficiently solved by the code bvptwp.m/twpbvplc.f that uses a deferred correction scheme based on Lobatto formulae [3]. The results obtained by applying the code to both two- and three-dimensional problems describing very involved scenarios, show the effectiveness of the whole procedure.

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On the computation of fractional power of operators

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We consider the numerical approximation of $\mathcal{L}^{-\alpha}$, $0 < \alpha < 1$, where $\mathcal{L}$ is an accretive operator acting on a separable Hilbert space $\mathcal{H}$, with numerical range contained in a sector of the complex plane symmetric with respect to the real axis. This problem finds immediate application when solving equations involving a fractional diffusion term like $(-\Delta)^\alpha$ where Δ denotes the standard Laplacian. In this case the operator is self-adjoint.

By exploiting the existing representations of the function $\lambda^{-\alpha}$ in terms of contour integrals (see [1, 6]), after suitable changes of variable and quadrature rules one typically finds rational approximations of the type

$$\mathcal{L}^{-\alpha} \approx \mathcal{R}_{n-1,n}(\mathcal{L}), \quad \mathcal{R}_{n-1,n}(\lambda) = \frac{p_{n-1}(\lambda)}{q_n(\lambda)}, \quad p_{n-1} \in \Pi_{n-1},\, q_n \in \Pi_n,$$

where n is equal or closely related to the number of points of the quadrature formula. In this work we present a comparative analysis of the most reliable existing method based on quadrature rules, with particular attention to the error estimate and the asymptotic rate of convergence (see e.g. [2, 3, 4, 5]). The analysis is given in the infinite dimensional setting, so that all results can be directly applied to the discrete case, independently of the discretization used.

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An extension of nonstandard finite differences with application to a vegetation model

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Discretization schemes based on NonStandard Finite Differences (NSFD) are a modification of Standard Finite Differences (SFD) schemes in which the classical denominators Δt and Δx (and also the related powers, if present) are replaced by particular scalar *denominator functions* satisfying certain conditions. Furthermore, some terms of the SFD schemes can be approximated in the NSFD methods with *non-local representations* (see, e.g., [2] and [7]). The goal of these techniques is to improve the stability of SFD schemes built for the solution of ordinary and partial differential equations, being also able to preserve the positivity and the equilibrium points properties of the continuous model.

In this talk [3], we extend the classical NSFD methodology by allowing the use of non-scalar denominator functions, inspired by Time-Accurate and highly-Stable Explicit operators (see [1]), also showing the connections between NSFD and exponentially fitted numerical methods (see, e.g, [4] and [6]), thanks to which it is possible to preserve the oscillation frequency, if a-priori known, of the exact continuous model solution. Finally, we apply the generalized NSFD methodology to a vegetation linear-diffusion non-linear-reaction model [5], showing through numerical tests the advantages of the proposed numerical technique.

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Non-periodic spectral method for a nonlinear peridynamic model

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In the framework of nolocal continuum mechanics, peridynamics is a non-local theory able to capture singularities and fractures without using partial derivatives. We focus on a one-dimensional nonlinear model of peridynamics and propose a spectral method based on the Fourier and Chebyshev polynomials to discretize in space. The main capability of the method is that it avoids the assumption of periodic boundary condition in the solution and can benefit of the use of the fast Fourier transform (FFT).

This is a joint work with Luciano Lopez from Università degli Studi di Bari Aldo Moro

Numerical solution of the fractional diffusion equation by spline approximations

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In recent years, fractional differential equations have occupied a prominent place in modeling anomalous diffusion, i.e., transport processes where long-range correlations between particles or anomalous long jumps of particles can occur. Anomalous diffusion has been observed in a variety of materials, such as porous media, biological tissues, condensed matter [6]. At the same time, the demand of efficient numerical methods for solving diffusion differential problems with fractional derivatives has increased enormously.

In this talk, we present a numerical method suitable to solve fractional diffusion problems. For the time derivative we consider the Caputo derivative since it retains many peculiar features of the classical derivative [1]. As for the space derivative, we use the symmetric Riesz-Caputo derivative since it is more suitable to model transport processes in which contributions from both sides of the spatial domain have to be taken into account.

To solve the differential problem, we approximate its solution by a spline expansion [5], whose coefficients are evaluated by a collocation method [2, 3]. Spline approximations of operator equations can be efficiently evaluated exploiting the explicit expression of both classical and fractional derivatives of the spline basis [5, 3, 4]. We present some numerical tests to show the performance of the proposed method.

This is a joint work with E. Pellegrino and C. Sorgentone.

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Runge-Kutta schemes for the numerical solution of linear inhomogeneous IVPs

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Runge-Kutta methods for the numerical solution of inhomogeneous linear initial value problems with constant coefficients [1], [2] is considered.

A general procedure to construct explicit s-stage RK methods with general order s depending on the nodes $c_i, i = 1, \ldots, s$ is presented. This procedure only requires the solution of successive linear equations in the elements of the matrix $\mathbf{A}$ and avoids the solution of non linear equations.

Finally, we present several RK schemes with number of stages $s = 5, \ldots, 8$ and maximal order $p = s$ for the class of problems under consideration.

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On the destabilisation linear stochastic differential systems with non-normal drift

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In the ODEs theory, it has been wide deeply studied the effect of highly non-normality on linear systems of ordinary differential equations with a matrix of constant coefficient. In this talk, we present a particular instance of the general problem presented by Higham and Mao in [4] of destabilising a non-normal linear homogeneous system by a noisy term. We analytically construct a mean-square destabilising perturbation when the dimension of the system goes to infinity and the matrix of the cofficients assumes a particular bidiagonal bidiagonal form, which represents a prototype of a strongly non-normal case. Finally, we explore the numerical counterpart of the problem, analyzing the corresponding behaviour of the numerical stablity matrices of the stochastic θ-methods.

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Potentiality of the code HOFiD_bvp in solving different kind of second order bounday value problems

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Many applications in engineering, chemistry, physics and biology give rise to singularly perturbed boundary value problems, also with singularity, of second and high order. These kind of problems can be really stiff by the choice of perturbation parameters extremely strong. The aim of this talk is to show the numerical results obtained by using the Matlab code HOFiD_bvp in solving different kind of problems, starting from singular perturbation problems to singular problems, singular perturbation problems with a discontinuous source term and multipoint second order boundary value problems. The code is based on the application of the HOFiD methods [1, 2], using high order finite difference schemes with upwind to solve this class of BVPs. Code efficiency is guaranteed by implementation of the deferred corrections technique [3] and a mesh selection strategy based on the error equidistribution [4]. For the solution of nonlinear problems improvements on the local convergence of Newton method is considered and a continuation strategy is also available. All theoretical and numerical aspects, such as convergence and error estimation, will be described to emphasize accuracy of the numerical schemes and code potentiality.

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Matrix-Oriented Discretization Methods for Evolutionary Problems

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An interesting class of evolutionary problems is given by reaction-diffusion PDE systems where the coupling between diffusion and nonlinear kinetics can lead to the so-called Turing instability. In this case, a variety of spatial patterns can be attained as stationary solutions for longtime integration. To capture the morphological peculiarities of the Turing patterns, a very fine space discretization may be required, limiting the use of standard (vector-based) ODE solvers in time because of excessive computational costs.

We show that the structure of the diffusion matrix can be exploited to build matrix-oriented (MO) versions of some classical time integrators. In particular, we consider finite differences on square domains and classical Lagrangian FEM on x-normal domains and even on special surfaces. In the first case, the discrete problem is then reformulated as a sequence of Sylvester matrix equations, that we solve by the *reduced approach* in the associated spectral space [1, 3]. On general domains, at each time step, *multiterm Sylvester matrix equations* must be solved, where the additional terms account for the geometric contribution of the domain shape. In this case, we solve the matrix equations by the matrix-oriented form of the Preconditioned Conjugate Gradient (MO-PCG) method [2].

We illustrate our findings for the reaction-diffusion DIB model describing metal growth during battery charging processes. We apply the MO-IMEX Euler scheme for the approximation of stationary Turing patterns on rectangular domains and cylindrical surfaces. We will show encouraging results in terms of execution times and memory storage. For this reason, we present some initial recent results also for the efficient approximation of oscillatory solutions, like spiral waves and Turing-Hopf patterns, by some MO-splitting schemes.

This work is based on joint research with Maria Chiara D'Autilia, Massimo Frittelli (Università del Salento), Fasma Diele (IAC-CNR, Bari) and Valeria Simoncini (Università di Bologna).

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Pseudospectral approximation of characteristic roots of equations with infinite delay

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Delay equations with infinite delay (iDEs) are widely used in mathematical biology, and, in this context, the interest is often focused on the stability of equilibria as well as on their bifurcation analysis. But iDEs generate infinite-dimensional dynamical systems and so some numerical methods are needed. The pseudospectral discretization (PSD) has been successfully applied for the numerical stability analysis of equilibria, and for the numerical bifurcation in various contexts. Here we consider delay differential and renewal equations with infinite delay, for which the principle of linearized stability ensures that the stability of an equilibrium can be inferred from the eigenvalues of the infinitesimal generator of the linearization at the equilibrium [1]. By introducing a general abstract framework that encompasses both types of equations, we consider the PSD approach based on exponentially weighed polynomial interpolation at near-optimal Laguerre zeros. Some numerical tests illustrate the convergence of the characteristic roots for linear iDEs and the effectiveness of the technique for the bifurcation analysis of nonlinear equations [2, 3].

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Efficient Approximation of Singular BVPs in ODEs

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We deal with boundary value problems for systems of ordinary differential equations with singularities. Typically, such problems have the form

$$z'(t) = F(t, z(t)), \quad t \in (0, 1], \quad B_0 z(0) + B_1 z(1) = \beta,$$

where $\lim_{t\to 0} F(t, z(t)) = \infty$ and $\lim_{t\to 0} \partial F(t, z)/\partial z = \infty$. The analysis is usually done for the model equation

$$z'(t) = \frac{1}{t^\alpha} M z(t) + f(t, z(t)), \quad t \in (0, 1], \quad B_0 z(0) + B_1 z(1) = \beta,$$

where $f(t, z)$ may also be in the form of $g(t, z)/t$ with a smooth function $g(t, z)$. For $\alpha = 1$ the problem has a *singularity of the first kind*, while for $\alpha > 1$ the singularity is commonly referred to as *essential singularity*. We briefly recapitulate the analytical properties of the above problems with a special focus on the most general boundary conditions which guarantee their well-posedness.

To compute the numerical approximation for z we use polynomial collocation, because the method retains its high convergence order even in case of singularities. The usual high-order superconvergence at the mesh points does not hold in general. However, the uniform superconvergence is preserved (up to logarithmic factors). We will discuss how the collocation performs for problems with the inhomogeneity of the form $g(t, z)/t$.

The updated version of the MATLAB code `bvpsuite1.1` with the special focus on the above problem class has been implemented. For higher efficiency, estimate of the global error and adaptive mesh selection are provided. The code can be applied to arbitrary order problems in implicit form. Also systems of index 1 differential-algebraic equations (DAEs) are in the scope of the code. We illustrate the performance of the software with a special focus on parameter-dependent problems by means of numerical simulation of models in applications.

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On the minimum growth rate of solutions to linear switched systems

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We deal with discrete-time *linear switched systems* of the form

$$x(n+1) = A_{\sigma(n)}\, x(n), \quad \sigma : \mathbf{N} \longrightarrow \{1, 2, \ldots, m\},$$

where $x(0) \in \mathbf{R}^k$, the matrix $A_{\sigma(n)} \in \mathbf{R}^{k\times k}$ belongs to a finite family $\mathcal{F} = \{A_i\}_{1\leq i\leq m}$ and σ denotes the *switching law*.

It is known that the *most stable switching laws* are associated to the so-called *spectrum-minimizing products*, that is those products $P = A_{i_1}A_{i_2}\cdots A_{i_k}$ whose average spectral radius $\rho(P)^{1/k}$ equals the *lower spectral radius* $\check{\rho}(\mathcal{F})$ of the family $\mathcal{F}$. For families $\mathcal{F}$ sharing an invariant cone K, in this talk we show how to provide lower bounds to $\check{\rho}(\mathcal{F})$ by a suitable adaptation of the Gelfand limit in the framework of *antinorms* (Guglielmi & Z. [3]).

Then we briefly consider families of matrices $\mathcal{F}$ that share an invariant *multicone* K_{mul} (Brundu & Z. [1, 2]) and mention some generalizations of the known results on antinorms to this more general setting (Guglielmi & Z. [4]).

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S6. Numerical Linear Algebra and Applications

This session presents new developments and techniques of Numerical Linear Algebra (NLA), and discusses several areas in science and engineering in which NLA are applied. Applications that will receive particular attention include the solution of ill-posed problems, including inverse electromagnetic problems and image restoration, as well as network analysis.

Organizers:
Michela Redivo Zaglia, University of Padua
Giuseppe Rodriguez, University of Cagliari

Stein and Rosenberg, Perron and Frobenius

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The Stein-Rosenberg theorem on the convergence of the Jacobi and Gauss-Seidel methods for solving systems of linear equations is well known. It shows that both methods are simultaneously convergent or divergent and compares their speeds of convergence. In this talk, we first remind it, discuss its proofs and its history. Its proofs are based on the Perron-Frobenius theorem on the dominant eigenvalue of a nonnegative irreducible matrix. Its genesis will also be reminded.

Then, we give a brief account of the lives and the works of Stein and Rosenberg, and those of Perron and Frobenius.

The material of this talk is issue from [1].

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Fast Alternating Direction Multipliers Method by Generalized Krylov Subspaces

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The Alternating Direction Multipliers Method (ADMM) is a very popular and powerful algorithm for the solution of many optimization problems. In the recent years it has been widely used for the solution of ill-posed inverse problems. However, one of its drawback is the possibly high computational cost, since at each iteration, it requires the solution of a large-scale least squares problem.

In this talk we propose a computationally attractive implementation of ADMM, with particular attention to ill-posed inverse problems. We significantly decrease the computational cost by projecting the original large scale problem into a low-dimensional subspace by means of Generalized Krylov Subspaces (GKS). The dimension of the projection space is not an additional parameter of the method as it increases with the iterations. The construction of GKS allows for very fast computations, regardless of the increasing size of the problem. Several computed examples show the good performances of the proposed method.

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Graph Laplacian in $\ell^2 - \ell^q$ regularization for image reconstruction

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he use of the Laplacian of a properly constructed graph for denoising images has attracted a lot of attention in the last years. Recently, a way to use this instrument for image deblurring has been proposed in [1].

In this talk, we consider the $\ell^2 - \ell^q$ regolarization method, $0 < q < 2$, for image reconstruction with application in computer tomography and image deblurring [2]. Using the majorization-minimization method, we reduce to the minimization of a quadratic functional, whose solution is approximated in a subspace of fairly small dimension. Thanks to the projection into properly constructed subspaces of small dimension, the proposed algorithm can be used for solving large scale problems. Moreover, the projected problem can be also used for estimating the regularization parameter by the generalized cross validation or the discrepancy principle. Some numerical results compare our proposal with total variation and sparse wavelets reconstructions.

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Modular-proximal operators for adaptive regularization

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In this talk, structured optimisation problems of kind $\arg\min_{x\in X} f(x,y)+\lambda g(x)$ are considered as general form of Tikhonov-like functionals, where $f : X \times Y \longrightarrow \mathbb{R}$ represents a smooth convex fidelity term between the data $y \in Y$ and the solution $x \in X$, $g : X \longrightarrow \mathbb{R}$ is a proper, l.s.c., (possibly non-smooth) convex penalty term, and $\lambda > 0$ is the regularization parameter. In our approach, X and Y are both unusual variable exponent Lebesgue spaces $L^{p(\cdot)}$, that is, Lebesgue spaces where the exponent is not a constant value, but rather a function of the position of the domain [2, 3]. Due to their intrinsic space-variant geometrical properties, such Banach spaces can be naturally used for defining adaptive algorithms for the solution of ill-posed inverse problems.

For this purpose, we propose a proximal gradient algorithm in the (dual space of) $L^{p(\cdot)}$, where the proximal step is defined in terms of the modular function

$$\rho_{p(\cdot)}(x) := \int_\Omega \frac{1}{p(t)}|x(t)|^{p(t)}\mathrm{d}t,$$

which, thanks to its separability, allows for an efficient computation of the algorithmic forward-backward type iteration

$$x^{k+1} = \arg\min_{x\in L^{p(\cdot)}} \rho_{p(\cdot)}(x-x^k) + \lambda_k\langle \nabla f(x^k), x\rangle + \lambda_k g(x).$$

Convergence in function values is proved, with convergence rates depending on problem/space smoothness [5]. To show the effectiveness of the proposed modelling, some numerical tests highlighting the flexibility of the space $L^{p(\cdot)}$ are shown for exemplar signal and image deconvolution with mixed noise removal problems [5, 1, 4].

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Orthogonal Cauchy-like matrices

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A matrix $C \in \mathbb{R}^{n\times n}$ is Cauchy if its entries C_{ij} have the form

$$C_{ij} = \frac{1}{x_i - y_j}, \qquad i, j = 1, \ldots, n,$$

where x_i, y_j for $i, j = 1, \ldots, n$ are mutually distinct real numbers. Besides to their pervasive occurrence in computations with rational functions, Cauchy matrices play an important role in deriving algebraic and computational properties of many relevant structured matrix classes [1]. Indeed, they occur as fundamental blocks (together with trigonometric transforms) in decomposition formulas and fast solvers for Toeplitz, Hankel, and related matrices, see e.g., [2].

The main result of this contribution is to provide a complete description of the set of orthogonal Cauchy-like matrices, that is, the orthogonal matrices $K \in \mathbb{R}^{n\times n}$ with entries

$$K_{ij} = \frac{a_i b_j}{x_i - y_j}, \qquad i, j = 1, \ldots, n.$$

Interest in these matrices arises from the paper [3], where orthogonal matrices obtained by scaling rows and columns of Cauchy matrices are needed in the design of allpass filters for signal processing purposes. We illustrate their relationships with secular equations, the diagonalisation of symmetric quasiseparable matrices and the construction of orthogonal rational functions with free poles. Moreover, we characterize matrix families that are simultaneously diagonalized by orthogonal Cauchy-like matrices.

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On some iterative methods for Fredholm integral equations

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This talk deals with the numerical solution of Fredholm integral equations of the second kind,

$$f(y) + \int_{\mathcal{D}} k(x,y) f(x) d\mu(x) = g(y), \quad y \in \mathcal{D},$$

where the kernel k and right-hand side function g are given, the function f is to be determined, and $d\mu(x)$ is a nonnegative measure supported on a bounded or unbounded domain $\mathcal{D} \subset \mathbb{R}$.

Several iterative methods based on averaged Gauss quadrature formulae [2, 3] are proposed for the computation of Nyström interplants and numerical tests are given to show the performance of such methods.

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Computing Gaussian quadrature rules with high relative accuracy

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The computation of n-point Gaussian quadrature rules for symmetric weight functions is considered in this talk [1, 2]. It is shown that the nodes and the weights of the Gaussian quadrature rule can be retrieved from the singular value decomposition of a bidiagonal matrix of size $n/2$. The proposed numerical method allows to compute the nodes with high relative accuracy and a computational complexity of $\mathcal{O}(n^2)$. We also describe an algorithm for computing the weights of a generic Gaussian quadrature rule with high relative accuracy.

Numerical examples show the effectiveness of the proposed approach.

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A projection method for eigenvalue problems of linear nonsquare matrix pencils

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Consider the computation of all the finite eigenvalues of a linear matrix pencil $zB - A \in \mathbb{C}^{m\times n}$, $z \in \mathbb{C}$, A, $B \in \mathbb{C}^{m\times n}$ in a given simply connected open set $\Omega \subset \mathbb{C}$

$$A\boldsymbol{x} = \lambda B\boldsymbol{x}, \quad \boldsymbol{x} \in \mathbb{C}^n \setminus \{\mathbf{0}\}, \quad \lambda \in \Omega \tag{7}$$

and the corresponding eigenvectors. In a class of eigensolvers, such as the Sakurai–Sugiura method [4] and the FEAST algorithm [3], a complex moment consisting of a resolvent filters out undesired eigencomponents and extracts the desired ones in a pseudo-random matrix. Thus, methods of this kind project a regular matrix pencil onto the eigenspace associated with eigenvalues in a prescribed region and give the eigenvalues and the corresponding eigenvectors of a regular matrix pencil. This study extends a projection method for regular eigenproblems [5, 1] to the singular nonsquare case [2]. The extended method involves complex moments given by the contour integrals of generalized resolvents associated with nonsquare matrices. We establish conditions such that the method gives all finite eigenvalues in a prescribed region in the complex plane. In numerical computations, the contour integral is approximated by a numerical quadrature, similarly to the regular case. Each quadrature point gives a least squares problem to solve, and it can be solved independently. The primary cost lies in the solutions of linear least squares problems that arise from quadrature points, and they can be readily parallelized in practice. Numerical experiments on large matrix pencils illustrate this method. The new method is more robust and efficient than previous methods, and based on experimental results, it is conjectured to be more efficient in parallelized settings.

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Perron communicability and sensitivity of multilayer networks

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Modeling complex systems that consist of different types of objects leads to multilayer networks, where nodes in the different layers represent different kind of objects. Nodes are connected by edges, which have positive weights. De Domenico et al. [2] describe how multilayer networks with a set of N nodes and L layers can be represented by a supra-adjacency matrix $B \in \mathbb{R}^{NL \times NL}$. It is the purpose of this talk to carry out an investigation that focuses on the sensitivity of the network communicability (see, e.g., [1], [3]) to perturbations in the multilayer network, by studying the sensitivity of the Perron root of B.

In case of layer-coupled multiplex networks, in which nodes in different layers are identified with each other, one has

$$B = \mathrm{diag}[A^{(1)}, A^{(2)}, \ldots, A^{(L)}] + \mathbf{1}_L \mathbf{1}_L^T \otimes I_N - I_{NL},$$

where $A^{(\ell)} \in \mathbb{R}^{N \times N}$, with $\ell = 1, 2, \ldots, L$, is the non-negative adjacency matrix associated with the graph for the ℓth layer. Here, $\mathbf{1}_L \in \mathbb{R}^L$ denotes the vector of all entries one and $\otimes$ the Kronecker product. Such particular structure of the supra-adjacency matrices associated with multiplexes is exploited in the relevant structured eigenvalue sensitivity analysis; see [4]. Finally, as in [5], the network analysis we carry out sheds light on which edge weights to make larger to increase the communicability of the network, and which edge weights can be made smaller or set to zero without affecting the communicability significantly.

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Nonlinear least-squares problems: a regularized approach for the large scale underdetermined case

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In this work, we will describe a regularized Gauss–Newton method for the computation of the minimal-norm solution to underdetermined nonlinear least-squares problems [1, 2]. The approximate solution of the iterative method is obtained from that of Gauss–Newton by adding a correction vector, and depends on two relaxation parameters which are automatically estimated. We will focus on medium and large scale problems. In this case, the iterative method projects each linearized step in a suitable Krylov space. Numerical experiments concerning imaging science will be presented to illustrate the performance of the method.

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A New Method for the Construction of Schur Stable Matrix Families

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This study is concerned with the construction of some Schur stable matrices families consisting of linear and convex combinations of matrices $A \in S_N$ and $B \in M_N(\mathbb{C})$, where $S_N = \{A \in M_N(\mathbb{C}) \mid |\lambda_i(A)| < 1\}$ and $M_N(\mathbb{C}) = \{A \mid N \times N,\ a_{ij} \in \mathbb{C}\}$. It is well known that, according to Lyapunov's theorem, a necessary and sufficient condition for the matrix A to be Schur stable is that the Lyapunov matrix equation $A^*HA - H + I = 0$ has a Hermitian and positive definite solution H [4]. On the other hand, the parameter ω which indicates the Schur stability and the quality of the matrices, is defined as $\omega(A) = \|H\| \geq 1$. If $\omega(A) < \infty$ then A is Schur stable, otherwise it is not [1, 3, 4]. Given a specified parameter $\omega^* (> 1)$, if $\omega(A) \leq \omega^*$ then the matrix A is ω^*-Schur stable. In this talk, matrix families $\mathcal{A}_1$ and $\mathcal{A}_2$ will be introduced by linear and convex combinations of matrices $A \in S_N$ and $B \in M_N(\mathbb{C})$, respectively. In addition, some theorems and results about the Schur stability of these families will be given. A new method based on the Schur stability parameter and the continuity theorems, which indicates the sensitivity of the Schur stability, will also be described [2, 5]. According to this method, intervals $\mathcal{R}_1$ and $\mathcal{R}_2$ are obtained, which guarantee the Schur stability and ω^*-Schur stability of the matrix families $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Finally, illustrative examples related to the subject will be given.

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Construction of a sequence of orthogonal rational functions

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Orthogonal polynomials are an important tool to approximate functions. Orthogonal rational functions provide a powerful alternative if the function of interest is not well approximated by polynomials.
Polynomials orthogonal with respect to certain discrete inner products can be constructed by applying the Lanczos or Arnoldi iteration to appropriately chosen diagonal matrix and vector. This can be viewed as a matrix version of the Stieltjes procedure. The generated nested orthonormal basis can be interpreted as a sequence of orthogonal polynomials. The corresponding Hessenberg matrix, containing the recurrence coefficients, also represents the sequence of orthogonal polynomials.
Alternatively, this Hessenberg matrix can be generated by an updating procedure. The goal of this procedure is to enforce Hessenberg structure onto a matrix which shares its eigenvalues with the given diagonal matrix and the first entries of its eigenvectors must correspond to the elements of the given vector. Plane rotations are used to introduce the elements of the given vector one by one and to enforce Hessenberg structure.
The updating procedure is stable thanks to the use of unitary similarity transformations. In this talk rational generalizations of the Lanczos and Arnoldi iterations are discussed. These iterations generate nested orthonormal bases which can be interpreted as a sequence of orthogonal rational functions with prescribed poles. A matrix pencil of Hessenberg structure underlies these iterations. We show that this Hessenberg pencil can also be used to represent the orthogonal rational function sequence and we propose an updating procedure for this case. The proposed procedure applies unitary similarity transformations and its numerical stability is illustrated.

S7. Operator Algebras and Functional Analysis Methods for Applications

This section is addressed to cover the following topics: Operator algebras and applications to quantum physics; Theory of bases and frames and applications to signal analysis; approximation methods for applications.

Organizers:

Paolo Boggiatto, University of Turin
Rosario Corso, University of Palermo
Camillo Trapani, University of Palermo

Reflection positive representations and Hankel operators in the multiplicity free case

Maria Stella Adamo

University of Tokyo (Japan)

In joint work with K.-H. Neeb and J. Schober, we studied reflection positive representations by using positive Hankel operators. In this talk, we will discuss our new approach, mainly in the case of the integers and the real line. We showed that for these groups positive Henkel representations produce reflection positive representations in the regular multiplicity free case.

Representation of Operators Using Fusion Frames

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To solve operator equations numerically, matrix representations are used employing bases [4] or more recently frames[2]. For finding the numerical solution of operator equations [5] a decomposition in subspaces is needed in many applications. To combine those two approaches, it is necessary to extend the known method of matrix representation to the utilization of fusion frames [1].

We investigate this representation of operators on a separable Hilbert space with Bessel fusion sequences, fusion frames and Riesz decompositions [3]. Let $\{W_i\}_{i\in I}$ be a family of closed subspaces of $\mathcal{H}$ and $\{w_i\}_{i\in I}$ be a family of weights, i.e. $w_i > 0, i \in I$. The sequence $W = (W_i, w_i)$ is called a fusion frame for $\mathcal{H}$ if there exist constants $0 < A_W \leq B_W < \infty$ such that

$$A_W\|f\|^2 \leq \sum_{i\in I} w_i^2 \|\pi_{W_i} f\|^2 \leq B_W\|f\|^2, \qquad (f \in \mathcal{H}).$$

For two fusion frames W and V we define a matrix representation not only in a canonical but also in an alternate way - taking the particular property of the duality of fusion frames into account.

We will give the basic definitions and show some structural results, like that the functions assigning the alternate representation to an operator is an algebra homomorphism. We give formulas for pseudo-inverses and the inverses (if existing) of such matrix representations. We apply this idea to Schatten-class operators. Consequently, we show that tensor products of fusion frames are frames in the space of Hilbert-Schmidt operators.

We show how this can be used for the solution of operator equations and link our approach to the additive Schwarz algorithm. We provide small proof-of-concept numerical experiments. Finally we show the application of this concept to overlapped convolution and the non-standard wavelet representation.

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Do properties of frame-related sequences spread in a scale of Hilbert spaces?

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Frames have been introduced and studied as a powerful alternative to Hilbert space bases and they allow a deep theory. Also, they are very important for applications e.g. in signal analysis and in physics. There exist many notions of frame-related sequences, generalizing the notion of basis in Hilbert space. Both in mathematics and physics it is natural to consider a full scale of spaces, and not only a single one. Then a question arises: do properties of frame-related sequences in a space of a scale of Hilbert spaces, such as either completeness or the property of being a (semi-)frame, spread in a scale of Hilbert spaces? It has been found [1] that the answer is not always affirmative. Sometimes it is either affirmative, as for completeness, or partially affirmative, as for the property of being a lower (upper) semi-frame which is kept in larger (smaller) spaces of the scale, or negative, as the property of being a frame: a sequence cannot be a frame for both two different Hilbert spaces of a certain scale of Hilbert spaces.

Joint work with: Peter Balazs and Hessam Hosseinnezhad

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Real Paley-Wiener theorems

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A Paley-Wiener theorem is a characterization, by relating support to growth, of the image of a space of functions or distributions under a transform of Fourier type. This relation comes in terms of a compact and convex set in which the support of the function or distribution is included. In fact, the growth of $\hat{f}$ on $\mathbb{C}^d$ enables to retrieve the convex hull of the support of f, but no more precise information can be obtained from it. In the last years, a new type of results called "real Paley-Wiener type theorems" has received much attention. The idea is to try to bypass this theoretical obstruction for the classical Paley-Wiener theorems to "look inside" the convex hull of the support. The word "real" expresses that information about the support of f comes from growth rates associated to the function $\hat{f}$ on $\mathbb{R}^d$ rather than on $\mathbb{C}^d$ as in the classical "complex Paley-Wiener theorems". This theory was initiated by Bang, and here we follow the approach of Andersen and Andersen-De Jeu, facing the problem from the opposite point of view: starting by a rapidly decreasing function f we try to get information on the support of $\hat{f}$, which could be non-compact or even non-convex.

In particular, in [4] we work in the space $\mathcal{S}_\omega(\mathbb{R}^d)$ of rapidly decreasing ultradifferentiable functions for a weight ω (if $\omega(t) = \log(1+t)$ then $\mathcal{S}_\omega$ is the classical Schwartz space $\mathcal{S}$) and obtain the radius $R_{\hat{f}}$ of the support of $\hat{f}$ (which may be also $+\infty$) in terms of the derivatives of f or the Wigner transform of f:

$$R_{\hat{f}} = \lim_{n\to+\infty} \left(\max_{|\alpha|=n} \left\| e^{\lambda\omega\left(\frac{x}{|\alpha|+1}\right)} f^{(\alpha)}(x) \right\|_{L^p} \right)^{1/n}, \qquad \forall\lambda \geq 0, 1 \leq p \leq +\infty,$$

or

$$R_{\hat{f}} = \lim_{n\to+\infty} \| |\xi|_\infty^n \operatorname{Wig} f(x,\xi) \|_{L^{p,q}}^{1/n}, \qquad 1 \leq p,q \leq +\infty,$$

where $|\xi|_\infty = \max_{1\leq j\leq d} |\xi_j|$, and for the support of f:

$$R_f = \lim_{n\to+\infty} \| |x|_\infty^n \operatorname{Wig} f(x,\xi) \|_{L^{p,q}}^{1/n}, \qquad 1 \leq p,q \leq +\infty.$$

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Periodic pseudodifferential operators and applications to Gabor frames

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A Gabor system $\mathcal{G}(g,a,b)$ is a sequence of the type $\{g_{hk} = e^{2\pi i\, bk\cdot x} g(x - ah)\}_{h,k\in\mathbb{Z}^d}$, with g measurable function on $\mathbb{R}^d$, $a, b > 0$.

$\mathcal{G}(g,a,b)$ is said to be a frame in $L^2(\mathbb{R}^d)$ if $A\|f\|^2_{L^2} \leq \sum_{h,k\in\mathbb{Z}^d} |(f, g_{h,k})|^2 \leq B\|f\|^2_{L^2}$, for some $A, B > 0$ and any $f \in L^2(\mathbb{R}^d)$. Gabor frames play an important role in signal processes.

A wide literature is devoted in finding conditions on the window g and the lattice parameters $a, b > 0$, which allow the corresponding Gabor system to be a frame in $L^2(\mathbb{R}^d)$ so that the Gabor operator $S_{g,g}f = \sum_{h,k\in\mathbb{Z}^d}(f, g_{hk})g_{hk}$ is invertible in $\mathcal{L}(L^2)$ and a reconstruction formula $f = \sum_{h,k\in\mathbb{Z}^d}(f, g_{hk})\gamma_{h,h}$ is available, with $\gamma = S^{-1}_{g,g}g$. To this respect the very basic assumption is that a and b are "small enough".

In this talk we introduce results of continuity and invertibility in $L^p(\mathbb{R}^d)$ for pseudodifferential operators with symbols $\sigma(x,\xi)$ periodic in both the variable, which allow us to obtain sufficient conditions for the invertibility of $S_{g,g}$.

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Quantum Operations on Conformal Nets

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Conformal Field Theories (CFT) in 1 or 1+1 spacetime dimensions admit several "axiomatic" (i.e. mathematically rigorous and model independent) formulations. In these axiomatic schemes one can ask the questions: given a theory A, how many and which are the possible extensions B¿A or subtheories B¡A of A? What are their properties? Answers to these questions may lead to new models and to classification results. Extensions are typically described in the language of tensor categories, while subtheories require different ideas and methods. In the talk, I will report on our analysis of subtheories in the Operator Algebraic formulation. We make use of families of unital completely positive (UCP) maps acting on the CFT. These maps generalize the ordinary automorphisms of the CFT, they are compact in the pointwise ultraweak operator topology, and in some cases they can be identified with the convex space of positive probability Radon measures on a compact hypergroup (a classical generalization of a compact group, canonically associated with the inclusion B¡A). In general, they suffice to describe all the possible conformal inclusions B¡A.

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Semi-Fredholm operators on self-dual Hilbert W*-modules

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We consider semi-C*-Fredholm operators on self-dual Hilbert modules over a W*-algebra. We prove that the index of W*-Fredholm operators is well defined, and we characterize almost invertible C*-operators on self-dual Hilbert W*-modules in terms of C*-Browder decompositions. Moreover, we show that the set of semi-C*-Fredholm and the set of semi-C*-Weyl operators on self-dual Hilbert W*-modules form a semigroup under the multiplication and that the set of proper semi-C*-Weyl operators on self-dual Hilbert W*-modules is open in the norm topology. Finally, we illustrate by examples how the proofs of these results can be used to extend some results from the classical operator theory on Hilbert spaces.

Regularity of global solutions of PDE via time-frequency methods

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We present some results, contained in [3], on regularity of linear partial differential operators with polynomial coefficients in non isotropic ultradifferentiable classes. The problem of regularity was first introduced by Shubin in the frame of Schwartz functions and tempered distributions; a linear operator $A : \mathcal{S}' \to \mathcal{S}'$ is said to be regular if the conditions $u \in \mathcal{S}'$, $Au \in \mathcal{S}$ imply that $u \in \mathcal{S}$. Shubin formulates an hypoellipticity condition (in his global pseudodifferential calculus), that is sufficient to have regularity of the correponding operator. On the other hand, such hypoellipticity is far to be necessary, as there are several examples of operators which are not hypoelliptic but are regular (such as the Twisted Laplacian). The problem of characterizing regularity for classes of operators is quite hard. Even in very particular cases (as for ordinary differential operators with polynomial coeffcients) necessary and sufficient conditions for regularity are not known. Various results have been obtained in this field, showing classes of Partial Differential Operators that are regular, both in the classical sense and in scales of ultradifferentiable spaces, cf. [1, 2, 4, 5].

In this work we study regularity of partial differential equations with polynomial coefficients in non isotropic Beurling spaces of ultradifferentiable functions of global type. We study the action of transformations of Gabor and Wigner type in such spaces and we prove that a suitable representation of Wigner type allows to prove regularity for classes of operators that do not have classical hypoellipticity properties.

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On the Arens-Michael decomposition of $CV_{(0)}(X, A)$

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The Arens-Michael decomposition has been a powerful tool to study complete m-convex algebras and their applications. If X is a completely regular Hausdorff space, V a Nachbin family on X and A a locally convex algebra, let $CV_{(0)}(X, A)$ be the function algebra of all weighted vector-valued continuous functions with the topology given by the uniform seminorms induced by V. In this talk we study the Arens-Michael decomposition of this algebra when it is complete and m-convex, and relate it to another projective limit given in terms of the factors of the decomposition of A.

Spectral wavelet packets frames for signals on finite graphs

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Classical transforms, as Fourier, wavelet, wavelet packets and time-frequency dictionaries have been generalized to functions defined on finite, undirected graphs, where the connections between vertices are encoded by the Laplacian matrix. The main goal is to obtain atoms which are jointly localized both in the vertex domain (the analogue of the time domain for signals on the real line) and the graph spectral domain (the analogue of the frequency domain).

Despite working in a finite and discrete environment, many problems arise in applications where the graph is very large, as it is not possible to determine all the eigenvectors of the Laplacian explicitly. For example, in the case of our interest: a voxel-wise brain graph $\mathcal{G}$ with 900760 nodes (representing the brain voxels), and signals given by the fRMI (functional magnetic resonance imaging).

We present a new method to generate frames of wavelet packets defined in the graph spectral domain to represent signals on finite graphs.

Joint work with Iulia Martina Bulai.

Scaling limits of lattice quantum fields by wavelets

Yoh Tanimoto

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We present a renormalization group scheme for lattice quantum field theories in terms of operator algebras. The renormalization group is considered as an inductive system of scaling maps between lattice field algebras, and lattice fields are identified with the continuum field smeared with Daubechies' scaling functions. We show that the inductive limit of free lattice ground states exists and extends to the vacuum state on the continuum field.

Riesz-Fisher maps, Semiframes and Frames in rigged Hilbert spaces

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Given a Hilbert space $\mathcal{H}$, a sequence of vectors $\{f_n\}$ in $\mathcal{H}$ is a frame if there exists $A, B > 0$ such that :

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f|f_n\rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

As known, frames are generalizations of orthonormal bases, and their versatility is the motivation of the crescent importance in applications (signal analysis, image processing...) and in various areas of pure mathematics (time-frequence analisis, sampling theory, ...). However, this framework in Hilbert space does not include the case of *generalized eigenvectors* $\{\omega_x\}_{x\in X}$ (i.e. eigenvectors of an essetially self-adjoint operator A on $\mathcal{D} \subset \mathcal{H}$, x varies in some measure space X) that does not belongs to $\mathcal{H}$, but that can be viewed as distributions.That is the case of eigenvectors of continuous spectrum in QM.This motivates the extension of frames and bases to a rigged Hilbert space, that is the triplet:

$$\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^{\times}$$

where $\mathcal{D}$ is a locally convex space continuously embedded in $\mathcal{H}$ and $\mathcal{D}^{\times}$ the conjugate dual of $\mathcal{D}$.

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S8. Operators in Function Spaces: convergence properties and applications

The proposed session focuses on the properties of families of operators in function spaces, in particular (but not limited to) convolution integral operators, linear or non linear Urysohn type operators, discrete operators like sampling series and their generalizations. The aim is to point to the state of current researches concerning convergence properties and their concrete applications (for example to signal analysis and image reconstruction).

Organizers:
Carlo Bardaro, University of Perugia
Ilaria Mantellini, University of Perugia

A Characterization of the Rate of the Simultaneous Approximation by Generalized Sampling Series

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We establish a direct and a matching two-term strong converse inequality in terms of moduli of smoothness for the rate of the simultaneous approximation of generalized sampling series and their Kantorovich modification in the $L_p, 1 \leq p < \infty$, and uniform norm on $\mathbb{R}$. They yield the saturation property and class of this approximation operator.

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Weighted Approximation for the Families of Sampling Operators

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In this work, we present an extension of approximation behaviours of generalized sampling series and the convergence of so-called sampling Kantorovich operators for functions in weighted spaces. In the first frame we prove a point wise and uniform convergence for the operators. And a rate of convergence by means of weighted modulus of continuity is established, a Voronovskaja type theorem is also obtained. In the second frame we prove quantitative estimates for the rate of convergence of the above mentioned operators. Lastly, pointwise convergence results in quantitative form by means of Voronovskaja type theorems have been given for both operators.

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Convolution operators in spaces of almost periodic functions and applications

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The theory of almost periodic functions initiated by Bohr nearly a hundred years ago has been widely developed. In particular, it is connected with the fact that such functions have applications in many areas. Let us indicate that almost periodic patterns, which correspond to almost periodic measures, describe the structure of quasicrystals. There are many various classes of almost periodic functions. In this talk we are going to focus mainly on almost periodic functions in view of the Lebesgue measure (briefly μ-a.p. functions). Let us emphasize that μ-a.p. functions possess more complex nature than classical Stepanov almost periodic functions.

In this talk we are going to present some properties of μ-a.p. functions with a particular emphasis on their behavior under convolution. As applications of our results we will present a theorem concerning μ-a.p. solutions to linear differential equations of the first order.

Finally, we will focus on convolutions of some particular μ-a.p. functions with some restrictions of exponential functions. We will discuss some topological as well as set-theoretical properties connected with those convolutions.

The results presented in this talk come from the papers [2] and [3]. Moreover, we refer the reader interested in basic properties of almost periodic functions and some of their perturbation (in particular, in those connected with the convolution) to the recently published monograph [1].

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Approximation properties of the sampling Kantorovich operators: regularization, saturation, inverse results and Favard classes in L^p-spaces

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In the present talk, a characterization of the Favard classes for the sampling Kantorovich operators based upon bandlimited kernels has been discussed ([1]).

In order to achieve the above result, a wide preliminary study has been necessary. First, suitable high order asymptotic type theorems in L^p-setting, $1 \leq p \leq +\infty$, have been proved. Then, the regularization properties of the sampling Kantorovich operators have been investigated. Further, for the order of approximation of the sampling Kantorovich operators, quantitative estimates based on the L^p modulus of smoothness of order r have been established. As a consequence, the qualitative order of approximation is also derived assuming f in suitable Lipschitz and generalized Lipschitz classes. Finally, an inverse theorem of approximation has been stated, together with a saturation result, allowing to obtain the desired characterization.

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Discretization in Generalized Function Spaces

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The operation "discretization" usually means that functions are mapped to sequences of real or complex numbers. These sequences can moreover be finite or infinite. It is clear, "discretization" cannot be applied to all kinds of functions and it stands outside of two different kinds of spaces, function spaces and sequence spaces. Furthermore, four different Fourier transforms are involved, the integral Fourier transform for integrable (non-periodic) functions and the finite Fourier transform for periodic (locally integrable) functions, on one hand, and the Discrete-Time Fourier Transform (DTFT) and the Discrete Fourier Transform (DFT) for infinite and finite sequences, on the other hand [1]. However, "discretization" can be treated in Schwartz' generalized function spaces [2], such as the space of tempered distributions, where it is an operation that maps tempered distributions onto tempered distributions and its Fourier transform is the Fourier transform on tempered distributions. Recently it has been shown that this Fourier transform reduces to the four, usually defined Fourier transforms [3]. The setting of tempered distributions moreover allows to show that discretization and periodization are Fourier transforms of one another and their inverses, regularization and localization, form another Fourier transform pair [4]. A generalization of this concept is to understand discretization, periodization, regularization and localization as a family of four operations whose members are related to one another by three kinds of reciprocity, (i) reciprocity with respect to multiplication, (ii) reciprocity between multiplication and convolution and (iii) reciprocity with respect to convolution. Another important family is integration, differentiation, Fourier-domain integration and Fourier-domain differentiation. The former is Woodward's operational calculus [5, 6] and the latter is Heaviside's operational calculus [7]. Both are intensively used today in electrical engineering.

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Frame recycling

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Grafakos and Sansing have shown how to obtain directionally sensitive time-frequency decompositions in $L^2(\mathbb{R}^n)$ based on Gabor systems in $L^2(\mathbb{R})$; the key tool is the "ridge idea," which lifts a function of one variable to a function of several variables [1]. We generalize their result by showing that similar results hold starting with general frames for $L^2((\mathbb{R})$, both in the setting of discrete frames and continuous frames. This allows to apply the theory for several other classes of frames, e.g., wavelet frames and shift-invariant systems. We will consider applications to the Meyer wavelet and complex B-splines. In the special case of wavelet systems we show how to discretize the representations using ϵ-nets [2]. We will close with a short discussion of partial ridges [3].

This is joint work with Peter Massopust (TU München), Ole Christensen (DTU Lyngby) and Florian Heinrich (University of Passau).

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Nonlinear composition operators in Grand Lebesgue spaces

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Let Ω be an open subset of $\mathbb{R}^n$ of finite measure. Let f be a Borel measurable function from $\mathbb{R}$ to $\mathbb{R}$. We prove necessary and sufficient conditions on f in order that the composite function $T_f[g] = f \circ g$ belongs to the Grand Lebesgue space $L_{p),\theta}(\Omega)$ whenever g belongs to $L_{p),\theta}(\Omega)$.

We also study continuity, uniform continuity, Hölder and Lipschitz continuity of the composition operator T_f in $L_{p),\theta}(\Omega)$.

Accurate computation of the multidimensional fractional Laplacian

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In this talk we discuss approximation formulas for the fractional Laplacian $(-\Delta)^{\alpha/2}$, $0 < \alpha < 2$, in the framework of the method approximate approximations. The fractional Laplacian appears in different fields of mathematics (PDE, harmonic analysis, semi- group theory, probabilistic theory) as well as in many applications (optimization, finance, materials science, water waves). If we introduce the convolution

$$\mathcal{N}_\alpha(f)(\mathbf{x}) = c_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n-2+\alpha}} d\mathbf{y}, \quad c_{n,\alpha} = \frac{2^{\alpha-2}}{\pi^{n/2}} \frac{\Gamma(\frac{n-2+\alpha}{2})}{\Gamma(\frac{2-\alpha}{2})}, \tag{8}$$

then the fractional Laplacian can be represented as the ordinary Laplacian of the volume potential $\mathcal{N}_\alpha f$,

$$(-\Delta)^{\alpha/2} f(\mathbf{x}) = -\Delta \mathcal{N}_\alpha(f)(\mathbf{x})\,. \tag{9}$$

We propose a method of an arbitrary high order for the approximation of $\mathcal{N}_\alpha f$ and $(-\Delta)^{\alpha/2} f$, $n \geq 3$, which is based on the approximation of the function f via the basis functions introduced by approximate approximations (cf. [2]), which are product of Gaussians and special polynomials. Then the n-dimensional integral (8) applied to the basis functions is represented by means of a one-dimensional integral where the integrand has a separated representation, i.e., it is a product of functions depending only on one of the variables.

This construction enables to obtain one-dimensional integral representations with separated integrand also for the fractional Laplacian (9), when applied to the basis functions. An accurate quadrature rule and a separated representation of the density f provide a separated representation for $\mathcal{N}_\alpha f$ and $(-\Delta)^{\alpha/2} f$. Thus, only one-dimensional operations are used and the resulting approximation procedure is fast and effective also in high-dimensional cases, and provides approximations of high order, up to a small saturation error. We prove error estimates and report on numerical results illustrating that our formulas are accurate and provide the predicted convergence rate $2, 4, 6, 8$ (cf. [1]).

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Convergence of norms and singular values of generalized Toeplitz matrices

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We consider sequences of matrices that generalize finite sections of Toeplitz operators. Using C*-algebras and limit operators techniques we obtain results that give the convergence of the norms and the convergence of singular values. Connections with Neural Networks will also be mentioned (ongoing research).

Part of the talk is based on joint work with B. Silbermann, [1].

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Perturbation by Weakly Continuous Forms

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In this talk we present a perturbation of a closed form by a weakly continuous form. The perturbation leads to a new semigroup whose difference with the given semigroup consists of compact operators. We apply the results to *elliptic operators* on the Hardy space and generalise a class of quasicontractive semigroups acting on Hardy and weighted Hardy spaces [1].

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Quantitative estimates for nonlinear sampling Kantorovich operators in functional spaces

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In real world applications, signals can be suitably reconstructed by nonlinear procedures; this justifies the study of nonlinear approximation operators. A wide literature can be found in [5, 1, 6, 4, 2, 3].
Herein, we present some quantitative estimates for the nonlinear sampling Kantorovich operators in the multivariate setting using the modulus of smoothness of $L^\varphi(\mathbb{R}^n)$. As a consequence, the qualitative order of convergence can be obtained, in case of functions belonging to suitable Lipschitz classes. The general frame of Orlicz spaces allows us to deduce the corresponding estimates in several instances of well-known and useful spaces, as L^p-spaces, Zygmund spaces and exponential spaces. Moreover, in the particular case of L^p-spaces, we also obtain a direct estimate that is sharper than that one achieved in the general case of Orlicz spaces, thanks to the properties of the modulus of smoothness in L^p.
Several examples of nonlinear multivariate sampling Kantorovich operators, by using some special kernels, are provided.

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Approximation properties of Durrmeyer-sampling type operators in functional spaces

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Sampling-type operators have been introduced in order to give an approximate version of the celebrated classical sampling theorem. Here, we have studied the Durrmeyer-Sampling type operators (DSO) [3] (see also [6, 2]), which represent a further generalization of the well-known Generalized and Kantorovich-Sampling operators [1, 5].
The talk is devoted to show some recent approximation results for DSO in the multidimensional frame, based mainly on the study of a modular convergence theorem in the general setting of Orlicz spaces [4]. This result implies also the convergence in remarkable particular cases, such as in L^p-spaces, Zygmund spaces and exponential spaces. Including also the case of not necessarily continuous functions, the above results turn out to be particularly useful in the applications, where most of the real world signals (such as digital images) are not represented mathematically by continuous functions.
For the sake of completeness of the theory, we have also provided a pointwise and uniform convergence theorem and some quantitative estimates.
Finally, several examples for different types of kernels will be discussed.

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Derivative plane sampling and weighted differential operator

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The Whittaker–type derivative plane sampling reconstruction formula was established about three decades ago by J. R. Higgins in [1]. The speaker confirmed Higgins' result by another method and extended it for the stochastic processes class $L^\alpha(\Omega, \mathfrak{F}, \mathsf{P});\ 0 \le \alpha \le 2$ in the α–mean and almost sure sense, when the input processes possess spectral representation. Here the (p, q)–order weighted differential operator's Whittaker–Higgins type reconstruction formula is established for entire functions coming from Leont'ev functions space $[2, \pi\psi/2], \psi > 0$, applying the circular truncation error's upper bound, which vanishes with exponential rate. Special cases are also presented.

Keywords: (p, q)–order weighted differential operator; Leont'ev spaces of entire functions; circular truncation error; derivative sampling; truncation error upper bounds; Weierstraß sigma–function; Whittaker–type plane sampling reconstruction.

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Types of convergence which preserve continuity

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It is well known that a locally uniformly convergent sequence of continuous functions always has a continuous limit function, whereas a pointwise convergent sequence of continuous functions may have a discontinuous limit.

This raises the question what additional assumption on top of pointwise convergence can be made in order to guarantee that the limit function of a sequence of continuous functions is always continuous. The first one who solved this problem was Arzelá in 1883, who introduced the notion of *quasi uniform convergence*.

In this talk we investigate - in addition to pointwise and locally uniform convergence - three further types of convergence in metric spaces, namely *quasi uniform*, *semi uniform* and *continuously uniform* convergence. We give criteria under which a sequence converges in one of these types and keep our eyes on those which preserve continuity.

In addition, we point out that several types of convergence can be used to characterize compactness.

As an application we use some of the theoretical results to the discussion of (autonomous) composition operators in the space BV of real-valued functions of bounded variation. There are known criteria guaranteeing that these operators map BV into itself. However, pointwise continuity of these operators is a delicate problem with an interesting history.

We present criteria under which sequences of composition operators converge locally uniformly and semi uniformly in the space BV. Moreover, we sketch a new and short proof of the pointwise continuity of such operators using semi uniform convergence.

Apart from recalling known and discussing new results we put a particular emphasis on examples and counter examples.

Sampling Kantorovich algorithm for the detection of Alzheimer's disease

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Among sampling-type operators, the Sampling Kantorovich operator represents a useful tool for dealing with discontinuous functions [2]. Its muldimensional version has been implemented and allows not only to reconstruct, but also to enhance the resolution of images, as it acts boths as a low-pass filter and as a magnifier, increasing spatial resolution of images [4]. Indeed, Sampling Kantorovich algorithm has been used, with satisfactory results, to both biomedical and engineering fields [1, 3].
The talk is focused on some recent results, which consist in the use of different algorithms, including Sampling Kantorovich algorithm, to process magnetic resonance images for the identification of biomarkers for Alzheimer's disease. The quality of reconstruction is evalueted, comparing the volumetric values of the images processed with the various algorithms, with the ground truth values, considered as reference. Moreover, the stereological Cavalieri\Point counting technique is used to infer volumetric data, starting from the knowledge of the planar sections.

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Weighted Approximation by Bivariate Generalized Sampling Series

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The present talk deals with weighted approximation of bivariate generalized sampling series. We present pointwise convergence of the series at continuity points of target functions and uniform convergence for weighted uniformly continuous functions. A rate of convergence for the series is also presented via bivariate weighted modulus of continuity and a Voronovskaja theorem for differentiable functions is obtained as well.

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Hardy spaces with variable exponents and maximal operators

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Let $p(\cdot): \mathbb{R} \to (0,\infty)$ be a variable exponent function satisfying the globally log-Hölder condition. We introduce the variable Hardy spaces $H_{p(\cdot)}(\mathbb{T})$ and $H_{p(\cdot)}[0,1)$ and give their atomic decompositions. It is proved that the maximal operator of the Fejér means of the Fourier series and Walsh-Fourier series is bounded on these spaces. This implies some norm and almost everywhere convergence results for the Fejér-means, amongst others the generalization of the well known Lebesgue's theorem.

A general method to study the convergence of nonlinear operators in Orlicz spaces

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We introduce a general setting in which we define nets of nonlinear operators whose domains are sets of functions defined in a locally compact topological group. We analyze the behavior of such net and detect the fairest assumption which are needed for the nets to converge with respect to the uniform convergence and in the setting of Orlicz spaces.

On Multi-dimensional Fractional Integral Transformations

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The subject of fractional integral transforms started in the early 1960's with the publication of Namias's paper on the fractional Fourier transfrom (The fractional order Fourier transform and its application to quantum mechanics. IMA J. Appl. Math. Vol. 25, No. 3, (1980)). Namias's paper caught the attention of many engineers and physicists who realized a number of applications of the transform in optics and signal processing. Prior to Namias's work, there were other related transforms that appeared in the work of N. Wiener and E. Condon but which went unnoticed for many years. More recently, several other fractional integral transforms, such as fractional Radon and fractional wavelet transforms were introduced in the literature. From another perspective, the fractional Fourier transform may be viewed as a special case of more general classes of integral transforms, such as the class of linear canonical transforms which was constructed in quantum mechanics as a class of unitary transformations acting in phase-space and as an integral representation of the metaplectic group. The extension of some of the classical concepts and notions, such as sam- pling theorms, convolution structures, uncertainty principles, to classes of these general transforms has been somewhat challenging, especially in higher dimensions. In this talk we will give a brief history of the subject and then discuss some recent developments on the extensions of some classical concepts to a general class of transformations.

S9. Orthogonal Polynomials, Interpolation and Numerical Integration

Orthogonal Polynomials and Interpolation are in the hard core of Approximation Theory, which is the basis for the development of numerical methods used in the solution of applied problems, such as the approximate calculation of (definite) integrals. The session will connect the background theory, Orthogonal Polynomials and Interpolation, with the practical application, Numerical Integration.

Organizers:

Sotiris Notaris, National and Kapodistrian University of Athens

Miodrag Spalević, University of Belgrade

Marija Stanić, University of Kragujevac

Christoffel-Darboux formula for orthogonal polynomials in several real variables

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Suppose that $\{p_n\}_{n=0}^{\infty}$ is a sequence of one variable real polynomials, which is orthogonal with respect to a Borel measure on $\mathbb{R}$. Then $\{p_n\}_{n=0}^{\infty}$ satisfies the three term recurrence relation, i.e.

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1}p_{n-1}(x), \quad n \geqslant 0,$$

with some $a_n, b_n \in \mathbb{R}$ (with $a_{-1} := 0$ and $p_{-1} := 0$). The Christoffel-Darboux formula is the equation:

$$\sum_{j=0}^{n} p_j(x)p_j(y) = a_n \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y}.$$

We are going to discuss a natural generalization of these formulas in the case of polynomials of several real variables. The three term recurrence relation is then the set of equations:

$$X_j Q_n \stackrel{V}{=} A_{n,j}Q_{n+1} + B_{n,j}Q_n + A^{\mathsf{T}}_{n-1,j}Q_{n-1}, \quad n \geqslant 0, \ j = 1, \ldots, d,$$

where $\{Q_k\}_{k=0}^{\infty}$ is a system of real orthogonal polynomials arranged in columns, where Q_k consists of polynomials of degree k; then $A_{n,j}$ and $B_{n,j}$ are real matrices of appropriate sizes. The notation "$\stackrel{V}{=}$" stands for "equality modulo an ideal V", which is inevitable, if we want to act in full generality (including e.g. polynomials orthogonal on a circle); this is a far-reaching refinement of results from [3, 4] published in [1]. The Christoffel-Darboux formula takes the form:

$$(x_j - y_j)\sum_{k=0}^{n} Q_k^{\mathsf{T}}(y)Q_k(x) \stackrel{V_2}{=} [A_{n,j}Q_{n+1}(y)]^{\mathsf{T}}Q_n(y) - Q_n(x)[A_{n,j}Q_{n+1}(y)]^{\mathsf{T}},$$

where $V_2 = V\otimes\mathcal{P}_d + \mathcal{P}_d\otimes V$ with $\mathcal{P}_d$ standing for the space of all polynomials in d variables (see [2]). Hopefully, the talk will be concluded with some examples (if time allows).

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Gaussian rule for integrals involving Bessel functions

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In this work we present a Gaussian type quadrature rule for the evaluation of integrals involving fractional powers, exponentials and Bessel functions of the first kind. In general, the technique commonly used in the computation of the coefficients of the three-term recurrence relation, for the corresponding orthogonal polynomials, is the Chebyshev algorithm (see [2, sect.2.3]). Nevertheless, it is well known (see e.g. [4]) that the computation of the recurrence coefficients can be inaccurate for growing number of quadrature points because the problem is severely ill conditioned. This issue can be partially overcame by using the modified moments (see [5], [2, sect.2.4], [6]), having at disposal a family of polynomials orthogonal with respect to a weight function similar to the one of the problem. This approach can be efficient in general but not always when working with unbounded intervals of integration (see [3] and [4]). In this framework, we present an alternative approach that is based on the preconditioning of the moment matrix. In particular, since the three-term recurrence coefficients can be written in terms of ratios of determinants of the moments matrix or slight modification of them (see [1, sect.2.7]), we exploit the Cramer rule to show that the coefficients can be computed by solving a linear system with the moment matrix. Since the weight function of the problem can be interpreted as a perturbation of the weight function of the generalized Laguerre polynomials, we use the moment matrix of these polynomials as preconditioner. The numerical experiments confirm the reliability of this approach and shows that it is definitely more stable than the modified Chebyshev algorithm.

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Internality of averaged Gaussian quadrature rules for modified Jacobi measures

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Averaged quadratures ([3, 4]) serve as a suitable method of approximating the error in the Gauss quadrature, but it is desirable that they have internal nodes. This is known to happen e.g. when the measure is one of the four Chebyshev measures modified by a linear divisor ([1, 2]). Now we investigate this question for analogous modifications of the Jacobi measures in general, when many of the required quantities are not known in explicit terms, and describe the exponents α and β for which it suffices to take the number n of nodes big enough.

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An iterative approach for a trigonometric Hermite interpolant

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In [2] it was introduced a trigonometric barycentric interpolant of an arbitrary 2π-periodic function in $[0, 2\pi)$ on some ordered nodes $0 \leq \theta_0 < \ldots < \theta_{n-1} < 2\pi$ which converges exponentially when the nodes are equidistant points or their images under a periodic conformal map [1] and has a logarithmic growth of the Lebesgue constant for a wide class of nodes [3]. We present here an iterative method to construct a trigonometric Hermite interpolant based on the latter interpolant. In fact, by using the auxiliary function

$$d_i(\theta) = 2\sin\left(\frac{x - x_i}{2}\right)$$

and the basis function $b_i(\theta)$ of the interpolant, it is possible to construct in an iterative way, similarly as done in [4] for the Floater-Hormann family of interpolant, the Hermite interpolant by considering

$$b_{i,j} = \frac{1}{j!} d_i(\theta)^j b_i(\theta)^{j+1}$$

and therefore the interpolant

$$r_j(\theta) = \sum_{i=0}^{n} \sum_{j=0}^{m} b_{i,j}(x) g_{i,j}$$

where

$$g_{i,0} = f(\theta_i) \qquad g_{i,j} = f^{(j)}(\theta_i) - r_{j-1}^{(j)}(\theta_i).$$

Furthermore, to implement it numerically we compute the differential matrix of the resulting interpolant at each iteration.

Finally, we are going to present some numerical tests.

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Error Estimates for Certain Quadrature Formulae

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Here we studied the error bound of Gauss-Legendre quadrature for analytic functions. The basic idea is to express the remainder of Gauss-Legendre quadrature as a contour integral, then the error bound is reduced to find the maximum of the kernel function:

$$K_n(z;\omega) = \frac{\varrho_n(z;\omega)}{\pi_n(z)}, \quad \varrho_n(z;\omega) = \int_{-1}^{1} \frac{\pi_n(t)}{z-t}\, dt, \quad z \in \mathbb{C} \setminus [-1,1]. \tag{10}$$

Inspired by the work of [1] and applying the results of [2], we obtained explicit and asymptotic formula of the kernel function $K_n(z;\omega)$ as $\rho \to \infty$. Explicit expression is used for determining location on the ellipses where maximum of the modulus of the kernel is attained.

Keywords: Gauss quadrature formulae, Legendre polynomials, remainder term for analytic function, error bound

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Some new results concerning the classical Bernstein cubature formula

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We present a solution to the approximation problem of the volume obtained by the integration of a bivariate function when a double integral cannot be computed exactly. The approximation of various double integrals can be done by a few cubature formulas (for instance, the Newton-Cotes cubature formulas) according to the specialty literature. Constructed by means of the bivariate Lagrange polynomial, trapezoidal and Simpson cubature formulas use a fixed number of nodes, resulting in a single possible approximation for a double integral. In order to be more flexible with this fact, we bring to the light a cubature formula constructed on the base of the classical bivariate Bernstein operator. As a valuable tool to approximate any volume resulted by integration of a bivariate function, we use the classical Bernstein cubature formula

$$\int_a^b \int_c^d F(x,y)dxdy \approx \frac{(b-a)(d-c)}{(n_1+1)(n_2+1)} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} F\left(a+\frac{k_1(b-a)}{n_1}, c+\frac{k_2(d-c)}{n_2}\right),$$

obtained as a continuation of our sustained research in [1], [2] and [3]. If the bivariate interval $[a,b]\times[c,d]$ ((the bivariate symmetrical interval $[-a,a]\times[-a,a]$) is large, then the classical composite Bernstein cubature formula is suitable for the approximation of a double integral. Numerical examples are given to increase the validity of the theoretical aspects.

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Weighted averaged Gaussian quadrature rules for modified Chebyshev measure

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The averaged and optimal averaged quadrature rules ([4, 7]) are a convenient method of approximating the error in the Gauss quadrature rule. However, to be fully applicable, they need to have internal nodes. We investigate how a weighted averaged rule for modified Chebyshev measure should be set in order to secure internality. The results are illustrated by numerical examples comparing the corresponding errors.

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Anti-Gaussian quadrature formulae of Chebyshev type

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Laurie (cf. [3]), in 1996, in an attempt to estimate practically the error of the Gauss quadrature formula, developed the anti-Gaussian quadrature formula, which is an $(n+1)$-point interpolatory formula designed to have an error precisely opposite to the error of the Gauss formula for all polynomials of degree up to $2n+1$. The anti-Gaussian formula enjoys nice properties: Its nodes interlace with the Gauss nodes and, with the possible exception of the first and the last one, they are contained in the support interval; its weights are all positive; and the formula has precise degree of exactness $2n-1$ and it can easily be constructed.

A Chebyshev type quadrature formula is an n-point interpolatory formula having equal weights, real nodes and degree of exactness (at least) n (cf. [2]). Equally-weighted quadrature formulae are useful in practice, because they minimize both the number of computations involved and the effect of random errors in the function values (cf. [1], Chapter 9). Furthermore, the study of Chebyshev type formulae is an intriguing mathematical problem.

In this talk, we examine whether there are positive measures admitting anti-Gaussian formulae of Chebyshev type.

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On the Gauss-Kronrod quadrature formula for a modified weight function of Chebyshev type

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In this paper, we consider the Gauss-Kronrod quadrature formulas for a modified Chebyshev weight. Efficient estimates of the error of these Gauss–Kronrod formulae for analytic functions are obtained, using techniques of contour integration that were introduced by Gautschi and Varga. Some illustrative numerical examples which show both the accuracy of the Gauss–Kronrod formulas and the sharpness of our estimations are displayed. Though for the sake of brevity we restrict ourselves to the first kind Chebyshev weight, a similar analysis may be carried out for the other three Chebyshev type weights; in the original paper, written in common by Miodrag Spalević, Ramon Orive, Ljubica Mihić and Aleksandar Pejčev, part of the corresponding computations are included in a final appendix.

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Spectral factorization of Laurent polynomials by means of quadrature formulas on the unit circle

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The spectral factorization of a Laurent polynomial $a(z)$ positive on the unit circle $|z| = 1$ consists in computing an algebraic polynomial $\gamma(z)$ such that $a(z) = \gamma(z)\gamma(z^{-1})$ and $\gamma(z^{-1})$ is Schur stable (all its roots in the open unit disk).

In this talk we deal with the computation of the spectral factor $\gamma(z)$ by means of its barycentric polynomial form. Uniformly distributed barycentric nodes on circles of radius less than one are considered. To obtain the barycentric form of $\gamma(z)$ we need to approximate certain integrals on the unit circle. We study the quadrature error. A bound of the absolute error $|\gamma(z) - \tilde{\gamma}(z)|$ of our computed approximation $\tilde{\gamma}(z)$. Numerical examples are given.

Optimal sets of quadrature rules in the Borges' sense for trigonometric polynomials

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In this article we consider the optimal sets of quadrature rules in the sense of Borges [1] for trigonometric polynomials of the both integer and semi-integer degree. Also, we consider the corresponding sets of quadrature rules when some of the nodes are fixed and prescribed in advance. In addition to the theoretical results, we will present the construction method and give appropriate numerical examples.

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Incorporating the external zeros of the integrand into certain quadrature rules

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Quadrature formulas are often constructed to be exact on the space of functions that are easily integrated and that are in some sense similar to the integrand. This motivates us to examine how the known properties of the integrand, such as its external zeros (zeros outside the (closed) interval of integration), can be used in order to improve the accuracy of certain quadrature formulas. In particular, we consider Gauss-type quadrature rules into which the external zeros of the integrand are incorporated.

S10. Positive Approximation Processes and Applications

This session aims to cover recent progresses in approximation of functions by positive linear operators in both finite and infinite-dimensional settings. Applications and connections with other fields are welcome, in particular with semigroup theory and evolution problems.

Organizers:

Octavian Agratini, Babeș-Bolyai University and Tiberiu Popoviciu Institute

Elena Berdysheva, University of Cape Town

Michele Campiti, University of Salento

Poisson approximation to the binomial distribution: extensions to the convergence of positive operators

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The idea behind Poisson approximation to the binomial distribution was used in [1] and subsequent papers in order to establish the convergence of suitable sequences of positive linear operators. The proofs in these papers are given using probabilistic methods. We use similar methods, but in analytic terms. In this way we recover some known results and establish several new ones. In particular, we enlarge the list of the limit operators and give characterizations of them.

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On Wachnicki operators

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The talk aims at a generalization W_α ($\alpha \geq -1/2$) of the Gauss-Weierstrass integral operator introduced by Eugeniusz Wachnicki [1]. The operator is intimately connected to a generalization of the heat equation.

It is defined as follows

$$W_\alpha(f; r, t) = \frac{1}{2t} \int_0^\infty r^{-\alpha} s^{\alpha+1} \exp\left(-\frac{r^2+s^2}{4t}\right) I_\alpha\left(\frac{rs}{2t}\right) f(s) ds,$$

where $(r,t) \in (0,\infty) \times (0,\infty)$ and I_α is the modified Bessel function of the first kind and fractional order α.

For $\alpha = -1/2$ the operator becomes the authentic Gauss-Weierstrass operator.

Our results focus on the asymptotic expansion of both W_α operators and their derivatives $\left(\frac{\partial}{\partial r}\right)^m W_\alpha$ of any order $m \in \mathbb{N}$.

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On optimal recovery problems in semi-linear metric spaces

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We consider optimal recovery problems for functions and integrals on classes of functions that take values in semi-linear metric spaces (L-spaces) and such that the functions themselves or their Hukuhara-type derivatives have a given majorant of the modulus of continuity. The recovery is made based on n values of the function or on the function's n mean values over intervals. We also obtain sharp Landau type inequalities and solve an analog of the Stechkin problem about approximation of unbounded operators by bounded ones and the problem of optimal recovery of an unbounded operator on a class of elements, known with error. A key role in obtaining our results is played by the generalization of the well-known Korneichuk–Stechkin lemma to the case of functions with values in L-spaces. The use of functions with values in L-spaces allows, in particular cases, to obtain results on optimal recovery of operators on classes of multi-valued, fuzzy-valued, and Banach-valued functions (in particular, random processes).

Metric Fourier approximation of set-valued functions of bounded variation

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We study set-valued functions (SVFs) mapping a real interval to compact sets in $\mathbb{R}^d$. Older approaches to the approximation of set-valued functions investigated almost exclusively SVFs with convex images (values). The standard methods suffer from convexification.

In this talk I will describe a new construction that adopts the trigonometric Fourier series to set-valued functions with general (not necessarily convex) compact images. Our main result is analogous to the classical Dirichlet-Jordan Theorem for real functions. It states the pointwise convergence in the Hausdorff metric of the metric Fourier partial sums of a set-valued function of bounded variation to a set determined by the values of the metric selections of the function. In particular, if the set-valued F is of bounded variation and continuous at a point x, then the metric Fourier partial sums of it at x converge to $F(x)$. If F is continuous in a closed interval, then the convergence is uniform.

Korovkin approximation of set-valued functions

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In 1988, Keimel and Roth [5] have established the analogous of the classical Korovkin theorem in the setting of cones of set-valued Hausdorff continuous functions. After this paper, many Korovkin-type results have been obtained even by introducing particular classes of monotone operators (see [1, 2]). In this talk we present different results obtained in the last years also in connection with the Korovkin approximation of vector-valued continuous functions [3].

We also consider cones of integrable set-valued functions and obtain the existence of Korovkin systems which may include integrable set-valued functions which are not Hausdorff continuous [4].

Some applications to classical sequences of Kantorovich and Bernstein-Durrmeyer type operators in the set-valued setting are also considered.

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On direct inequalities for the classical Bernstein and Szász-Mirakyan operators

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This talk deals with the approximation of continuous functions by the classical Bernstein operatos B_n and the Szász-Mirakyan operators S_t, in terms of the Ditzian-Totik modulus of smoothness ω_2^φ with the proper function φ.

Information about the rate of uniform convergence for both operators is given by the so-called direct inequalities,

$$\|B_n f - f\|_{[0,1]} \leq K_B \omega_2^\varphi \left(f; \frac{1}{\sqrt{n}} \right), \quad \varphi(x) = \sqrt{x(1-x)},$$

$$\|S_t f - f\|_{[0,\infty)} \leq K_S \omega_2^\varphi \left(f; \frac{1}{\sqrt{t}} \right), \quad \varphi(x) = \sqrt{x}.$$

Here, we focus on the absolute constants K_B and K_S. Asymptotic and non-asymptotic results are shown. We use a probabilistic approach, as well as a smoothing technique by considering approximants built from Steklov averages.

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Linear operators approximating discontinous functions. A probabilistic approach

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Many sequences of positive linear operators $\{L_n\}_{n\geq 1}$ allow for a probabilistic representation of the form $L_n(f,x) = Ef\left(x + \frac{Z_n(x)}{\sqrt{n}}\right), x \in I$, where I is a real interval, $f : I \to \mathbb{R}$ is any measurable function for which the preceding expectations exists, and $Z_n(x)$ is a random variable such that $x + Z_n(x)/\sqrt{n}$ takes values in I.

As an example, the classical Bernstein operators can be written as

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k} = Ef\left(x + \frac{Z_n(x)}{\sqrt{n}}\right), \quad x \in [0,1],$$

where $Z_n(x) = \frac{S_n(x)-nx}{\sqrt{n}}, x \in [0,1]$ and $S_n(x)$ is a random variable having the binomial distribution with parameters n and x. It is well known that for a bounded function f defined on $[0,1]$, $B_n(f,x) \to \frac{f(x-)+f(x+)}{2}$, where $f(x-)$ and $f(x+)$ are the left and right limits of f at x, respectively [3]. Rates of convergence in this case were provided by Bustamante et al.[2].

The main goal of this paper is to obtain linear operators acting on bounded functions on unbounded intervals such that $L_n(f,x) \to \alpha f(x+) + (1-\alpha) f(x-)$, with $0 \leq \alpha \leq 1$, giving at the same time rates of convergence. To do this, a probabilistic approach is used. As an illustration, we consider the sequence of linear operators $L_n^{(\alpha)} f(x) = Ef\left(x + \frac{Z_n^{(\alpha)}(x)}{\sqrt{n}}\right), x \in \mathbb{R}$, where $Z_n^{(\alpha)}(x)$ is a continuous random variable with probability density

$$\rho_\alpha(x) = \begin{cases} (1-\alpha)e^x & \text{if } x \leq 0 \\ \alpha e^{-x} & \text{if } x > 0 \end{cases} \qquad 0 \leq \alpha \leq 1.$$

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Voronovskaja type results for the Aldaz, Kounchev, Render modification of Baskakov type operators

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For $j > 1$, $j \in \mathbb{N}$ fixed and $n \geq j$, Aldaz, Kounchev and Render [1] introduced a modification of the Bernstein operator that preserves the monomials e_0 and e_j. This operator $B_{n,j} : C[0,1] \to C[0,1]$ is constructed using the classical original Bernstein basis functions $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $x \in [0,1]$ and is explicitly given by

$$B_{n,j}(f;x) = \sum_{k=0}^{n} f\left(t_{n,k}^{(j)}\right) p_{n,k}(x), \tag{11}$$

where

$$t_{n,k}^{(j)} = \left(\frac{k(k-1)\dots(k-j+1)}{n(n-1)\dots(n-j+1)}\right)^{1/j}.$$

We generalize the definition to Baskakov type operators and prove a corresponding Voronovskaja type result.

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The weighted Weierstrass Theorem for continuous functions defined on $[0, \infty)$ or on $(-\infty, \infty)$, proved using Bernstein-Chlodovski operators

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The Chlodovski extensions [3] of the classical Bernstein operators [2] were used in Kilgore [4] and [5] to prove the weighted versions of the classical Weierstrass Approximation Theorem, in the situation that the functions to be approximated are defined and continuous upon the interval $[0, \infty)$ using the weight $W(x) = e^{-x^\alpha}$ and satisfy $W(X)f(x) \to 0$ as $x \to \infty$. And in the similar case that interval is $(-\infty, \infty)$ using the weight $W(x)e^{-|x|^\alpha}$, and $W(x)f(x) \to 0$ as $|x| \to \infty$.

In each of the two above-described contexts, the Weierstrass theorem was already known to hold, but the new proofs were simple, basic in character, completely self-contained and autonomous. However, to approximate continuous functions defined upon $[0, \infty)$ it was necessary in constructing the new proof to assume that $\alpha > 1$, and for continuous functions defined on $(-\infty, \infty)$ one needed to assume that $\alpha > 2$.

Here, it is shown in each case above that the admissible value of α can be reduced. For the approximation on $[0, \infty)$ one may assume that $\alpha > \frac{1}{2}$. And the approximation on $(-\infty, \infty)$ requires $\alpha > 1$. As is known from [1] or [6], these are the least possible values of α for which the weighted version of the Weierstrass approximation can hold in the two respective situations. The proofs of these new results follow from minor changes to the Chlodovski operators.

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Iterative Shepard operator of least squares thin-plate spline type

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D. Shepard introduced in 1968 in [5] a very powerful method for approximating a given function f on a set of scattered data. A method for improving its accuracy was introduced by R. Franke and G. Nielson in [2].

T. Cătinaș and A. Malina introduced in [1] a new Shepard operator, based on the classical and the modified Shepard methods and the least-squares thin-plate spline function. The least squares thin-plate spline is defined for a point (x, y) and a set of nodes (x_i, y_i) as

$$F_i(x,y) = \sum_{j=1}^{i} C_j d_j^2 \log(d_j) + ax + by + c,\ i = 1, ..., N'$$

with $d_j = \sqrt{(x-x_j)^2 + (y-y_j)^2}$ and $C_j,\ a,\ b,\ c$ found such that they minimize

$$E = \sum_{i=1}^{N'} [F_i(x_i, y_i) - f(x_i, y_i)]^2,$$

for two different values of N', first with $N' = N$ interpolation nodes and second with $N' = k$ representative knot points, idea presented by J. McMahon in [4].

We propose an iterative modification of the Shepard operator of least squares thin-plate spline type, following an idea presented by A. Masjukov and V. Masjukov in [3]. The operator, denoted by u_L , is defined as

$$u_L(x,y) = \sum_{k=0}^{K} \sum_{j=1}^{N'} \left[u_{F_j}^{(k)} w\left((x-x_j, y-y_j)/\tau_k\right) / \sum_{p=1}^{N'} w\left((x_p - x_j, y_p - y_j)/\tau_k\right) \right],$$

where w is a continuously differentiable weight function with some particular properties, $u_{F_j}^{(k)}$ denotes the interpolation residuals at the kth step and τ_k is a scaling parameter.

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Some results in approximation theory by means of linear operators and generalized convergence

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In [4] P. Garrancho established general conditions to move different notions of generalized convergence to the setting of approximation theory by means of linear operators, and where the object of the approximation is certain generalized derivative of a function. In this talk, we show how we have continued the study, by facing to quantitative aspects. We also study the saturation class providing assuming that some asymptotic condition holds true. Finally, as applications of the previous results, we show how the notion of the recent weighted statistical convergence, due to Abdu Awel Adem and Maya Altinok [2], is a particular case of the generalized convergence analyzed here.

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On the Convergence of Series of Powers of Linear Positive operators

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Let $(L_n)_n$ be a sequence of linear positive operators $L_n : C_0[0,1] \to C[0,1]$ and denote $(L_n)^i = L_n \circ L_n \circ L_n \circ \ldots \circ L_n$ ($i \geq 0$ times). We consider the convergence of the sequences of operators of the form

$$A_n f = \sum_{i=0}^{\infty} Q_n(i)(L_n)^i f, \; f \in C_0[0,1],$$

where $Q_n(i) \in \mathbb{R}$ and $C_0[0,1]$ is a certain subspace of $C[0,1]$. We continue the previous study of geometric series of operators.

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New refinements of some inequalities

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Inequality Cauchy–Buniakowski–Schwarz and Aczél's inequality were studied by many mathematicians in their papers. We establish some new refinements of these, through the technique of monotony of a sequence.

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On Approximation Properties of some non-positive Bernstein-Durrmeyer Type Operators

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In this paper we shall introduce a new type of Bernstein Durrmeyer operators which are not positive on the entire interval $[0, 1]$. For these operators we will study the uniform convergence on all continuous functions on $[0, 1]$ as well as a result given in terms of modulus of continuity $\omega(f, \delta)$. A Voronovskaja type theorem will be proved as well.

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S11. Recent Advances in the Analysis and Numerical Solution of Evolutionary Integral Equations

Due to their ability to capture memory effects, integral and integro-differential equations of Volterra type describe a wide variety of dynamic processes that depend on the past history of the system. The effective impact of these models is evident in literature where analytical and numerical studies have, in recent years, led to a deeper understanding of many applications in the real world. Indeed, there have been very significant advances in the theory, applications and qualitative properties of both continuous and discrete solutions as well as in the development of numerically efficient methods. In this session, recognized researchers in the area of Volterra integral and related equations will present their recent achievements and discuss possible future developments.

Organizers:
Dajana Conte, University of Salerno
Teresa Diogo, CEMAT, University of Lisbon
Eleonora Messina, University of Naples "Federico II"

On the computation of the Wright function and its applications to Fractional Calculus

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The Wright function is defined by the following power series, convergent in the whole complex plane,

$$W_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\lambda n + \mu)}, \quad \lambda > -1,\, \mu \in \mathbb{C}. \tag{12}$$

Originally Wright assumed $\lambda \geq 0$ in connection with his investigation of the asymptotic theory of partition [6] and only in 1940 he considered $\lambda \in (-1, 0)$. The latter case is now referred to in the literature as *Wright function of the second kind* (WF2K) [7]. Although several representations of the Wright function have been introduced and many of its analytical properties have already been well studied (see, e.g., [2, 3, 4, 5]), its numerical computation is still an active research area.

In this talk we devote our attention to the numerical evaluation of WF2K, since this is the most interesting case for applications. We approach this topic by considering a technique based on the numerical inversion of the Laplace transform combined with a trapezoidal rule on a parabolic contour. We present some numerical experiments that validate both the theoretical estimates of the error and the applicability of the proposed technique to represent the solutions of fractional differential equations [1]. A code package that implements the algorithm proposed is contained in the repository: `github.com/Cirdans-Home/mwright`.

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Characterisation of the asymptotic behaviour of the mean–square of linear stochastic Volterra equations

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This talk concerns the asymptotic behaviour of autonomous stochastic Volterra equations of Itô–type. For simplicity, we will focus on scalar equations, but time permitting, finite dimensional equations will also be considered. The equation considered is

$$dX(t) = \int_{[0,t]} \nu(ds)X(t-s)\,dt + \int_{[0,t]} \mu(ds)X(t-s)\,dB(t), \quad t \geq 0; \quad X(0) = \xi, \tag{13}$$

where ν and μ are scalar finite measures on $[0,\infty)$, B is a standard one–dimensional Brownian motion, and ξ is a random variable independent of B which has a finite second moment. In this situation, there is a unique continuous adapted process which obeys (13), which we call the solution. The finiteness of the second moment of $X(0)$ is inherited by $X(t)$ for all $t \geq 0$. This is the so–called mean square of the solution, denoted $\mathbb{E}[X^2(t)]$, and in applications the long–run behaviour (as $t \to \infty$) is of great interest. Indeed, many sufficient conditions are known under which the mean square tends to zero as $t \to \infty$ (so called mean–square asymptotic stability), using developments of techniques familiar in the deterministic theory, including Liapunov functionals, Razumikhin's technique and so on. However, a characterisation in terms of the problem data of the mean–square asymptotic stability (or L^1 stability in mean square) has not been known until now.

In this talk, we develop a linear deterministic convolution Volterra integral equation for a functional of the solution, and a representation of the mean square of X in terms of that functional. Thus, the analysis of the long–run behaviour of the mean square becomes a problem in deterministic integral equations. However, the problem data in the resulting integral equation (i.e. the kernel and forcing function) are given indirectly in terms of the data of equation (13) via the resolvent r of the underlying deterministic equation, namely

$$r'(t) = \int_{[0,t]} \nu(ds)r(t-s), \quad t \geq 0; \quad r(0) = 1, \tag{14}$$

making the stability conditions hard to check.

Our main results show that it is possible (despite the fact that closed–form solutions of (14) are generally impossible to find) to give necessary and sufficient conditions on ν and μ such that the mean square tends to zero. Furthermore, the ideas involved allow stability characerisation results to be proven for perturbed equations of the form

$$dX(t) = \left(\int_{[0,t]} \nu(ds)X(t-s) + f(t)\right)dt + \left(\int_{[0,t]} \mu(ds)X(t-s) + g(t)\right)dB(t),$$
$$t \geq 0$$

where f and g are deterministic functions. Some extensions of the arguments allow a characterisation of the exponential asymptotic behaviour in the mean square, which is perhaps the most widely studied type of weighted stability in applications. In that situation, we are even able to develop a stochastic characteristic equation in terms of ν and μ, which identifies the dominant Liapunov exponent of the solution.

Approximating the fixed point of an affine operator

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We propose an algorithm for iteratively approximating the fixed point of a contractive affine operator. It is based on a perturbed version of the classic geometric series theorem, the error control that this provides, and the use of projections associated with certain Schauder bases. This is illustrated for a wide group of affine problems chosen for its great versatility, the linear Fredholm integral equations. Finally, we present some numerical examples in order to illustrate the behavior of the proposed method.

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Modeling metastatic tumor evolution, numerical resolution and growth prediction

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In this work we have introduced a generalized metastatic tumor growth model that describes the primary tumor growth by means of an Ordinary Differential Equation (ODE) and the evolution of the metastatic density using a transport Partial Differential Equation (PDE), [3]. The numerical method is based on the resolution of a linear Volterra integral equation (VIE) of the second kind, which arises from the reformulation of the ODE-PDE model, [2]. The convergence of the method is proved and error estimates are given. The computation of the approximate solution leads to solve well conditioned linear systems. Here we focus our attention on two different case studies: lung and breast cancer. We assume five different tumor growth laws, [1], for each of them, different metastatic emission rates between primary and secondary tumors, and last that the new born metastases can be formed by clusters of several cells.

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Highly accurate solution of fractional differential equations

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This talk deals with the numerical solution of nonlinear fractional differential equations (FDEs) of type

$$\begin{cases} D^{\alpha}y(t) = f(t, y(t)), & 0 \le t \le b, \\ y^{(i)}(0) = \gamma_i, & i = 0, \ldots, n-1, \end{cases}$$

where $n-1 < \alpha < n$, $n \in \mathbb{N}$, $\gamma_i \in \mathbb{R}$, $f : [0, b] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. The fractional derivative $D^{\alpha}y$ is the Caputo-type one [5]. FDEs arise in various fields, such as the dynamics in viscoelastic materials, the evolution of certain diseases, especially when the modelled phenomenon heavily depends on its past history.

On the side of the numerical simulation, solving FDEs with high accuracy is a challenging issue, since many numerical methods have low order of convergence. A powerful technique to obtain high order methods without increasing the computational cost consists of multistep collocation. We propose the class of two-step spline collocation methods [1, 2, 3], which double the order of convergence of the one-step collocation methods [4], at the same computational cost. In this talk, we analyze the convergence and stability properties of these methods, illustrate the main issues related to the implementation and finally show some numerical experiments.

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A posteriori error estimates for time discretization of abstract semi–linear fractional integro–differential equations

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The aim of this work is to provide a posteriori error estimates for time discretization of abstract semi–linear time fractional equations

$$\partial_t^\beta u(t) = Au(t) + F(u(t)), \qquad 0 < t \leq T, \tag{15}$$

where ∂_t^β stands for the time fractional derivative operator of order $\beta > 0$ in Riemann–Liouville sense, $1 < \beta < 2$, A is an abstract linear operator in a complex Banach space X, $A : D(A) \subset X \to X$, and $F(u)$ a reaction term under certain regularity conditions.

Our a posteriori error estimates (see [2]) are achieved through the maximal Hölder regularity of the analytical solution $u(t)$ to (15) in the context of θ–sectorial operators A, $0 < \theta < \pi/2$.

This approach has been previously considered for abstract ordinary differential equations, that is those where first time derivative is considered instead of fractional ones ∂_t^β (see [1]). In the fractional case the main difficulty arises from the lack of regularity typically occurs for solutions to differential equations involving time fractional derivatives and/or integrals.

Even though our work focused on providing estimates in a theoretical framework, throughout the work we show that all constants involved in the final estimates are in actual fact computables in the spirit of genuine a posteriori error esrtimates.

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Efficient computation of solutions of time-fractional diffusion-reaction equations

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Numerically solving time-fractional diffusion-reaction equations usually poses some not negligible challenges. Since derivatives of fractional order are non-local operators, their numerical treatment involves a persistent memory and, consequently, it is demanded a possible huge need for storage memory and computational resources.

On the basis of the recent work [1], we discuss a strategy to perform numerical simulations in an efficient way, namely by requiring a reasonable occupation of memory and an acceptable CPU time. This strategy is obtained by coupling an ImEx product-integration rule with a kernel compression scheme, a technique allowing the approximation of a non-local problem by a sequence of local problems.

The computational task required by the solution of a possible large number of linear systems of large size is further optimized by reformulating the difference scheme for the space operator in a matrix formulation (according to an approach recently proposed in [2]), so as to require the solution of Sylvester equations only with small matrices.

The accuracy of the proposed scheme is theoretically studied and validated by means of some numerical experiments and the efficiency of this strategy from the computational point of view is also verified.

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Numerical schemes for a class of singular fractional integro-differential equations

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Fractional derivatives and equations containing them have fascinated scientists for a very long time. Over the last few decades interest in the field has increased significantly because of new applications in Physics, Chemistry, Electrical Networks and so on [1, 3].

In [2], the unique solvability of singular fractional differential equations was studied. In the current talk we consider singular fractional integro-differential equations of the form

$$(D_0^\alpha M^\alpha u)(t) = \sum_{k=1}^{l} b_k (D_0^{\alpha_k} M^{\alpha_k} u)(t) + b(Vu)(t) + f(t), \quad 0 < t \leq T, \quad (16)$$

where the multiplication operator M^ν is defined by

$$(M^\nu u)(t) = t^\nu u(t), \quad 0 < t \leq T, \quad \nu \in \mathbb{R}, \quad u \in C[0,T],$$

V is a certain type of Volterra integral operator, $\alpha, \alpha_k, b, b_k, \in \mathbb{R}$, and

$$m < \alpha \leq m+1, \quad \alpha > \alpha_k \geq 0, \quad f \in C^m[0,T], \quad k = 1,2,\dots,l, \\ m \in \mathbb{N}_0 = \{0,1,2,\dots\}.$$

By $C^m[0,T]$ ($m \in \mathbb{N}_0$) we denote the space of m times continuously differentiable functions u on $[0,T]$; $C^0[0,T] = C[0,T]$. In equation (16) the fractional differential operator D_0^μ, of order $\mu \in [0,\infty)$, is defined as the inverse of the Riemann-Liouville integral operator

$$(J^\mu u)(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} u(s)ds, \quad u \in C[0,T], \quad t > 0, \ \mu > 0; \quad J^0 = I,$$

where I is the identity mapping and Γ the Euler gamma function.

In the talk we present some results about the unique solvability of equations of the form (16) and introduce a collocation based scheme for finding the numerical solution of such equations. We also give results of numerical experiments.

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Real exponential asymptotic behaviour is generic in the mean square of two-dimensional linear SDE's

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In this talk we are concerned with the mean square behaviour of linear 2-dimensional systems of stochastic differential equations with constant coefficients. The behaviour of such systems has been studied in great detail over the past fifty years and in particular a full characterisation of the mean square of n-dimensional systems of SDE's was given by Ludwig Arnold. Using Arnold's approach, one can show the mean square process obeys an $n^2 \times n^2$ system of ordinary differential equations, hence all information regarding the mean square process can be inferred by studying the eigenstructure of the associated $n^2 \times n^2$ coefficient matrix. For high dimensional problems finding exact evolutionary behaviour becomes highly intractable but for the case where $n = 2$, such computational issues can be surmounted and it is such eigenvalue analysis to which this talk is devoted. In particular we show that for arbitrarily small and large noise the dominant dynamics are **always** real exponential where dominance is classified by the long term behaviour of solutions of the mean square process. We also identify special cases wherein real exponential behaviour prevails independent of the level of noise introduced into the system. Our approach is to make a suitable coordinate transformation which results in a reduction of complexity in the characteristic polynomial of the associated 4×4 matrix. It should be noted we make no prior assumptions on structure of the underlying coefficient matrices to allow for a completely general treatment of the mean square. It is of particular interest that upon introduction of arbitrarily small noise into the underlying deterministic system, solutions are **unable** to produce oscillatory behaviour, even in the case when the solution of the noise-free equation has oscillatory solutions. The remarkable rarity of dominant oscillatory solutions, regardless of the eigenstructure of the drift and diffusion matrices in the two-dimensional case, leads to the conjecture that the phenomenon of dominant real exponential behaviour in the mean square may extend to arbitrarily many dimensions. However, the methods of proof used in this talk would not readily generalise to attack this conjecture. These are preliminary results in part of a larger study with Conall Kelly (UCC) on the seemingly generic appearance of dominant real exponential asymptotic behaviour in the mean square in autonomous linear stochastic equations.

A general collocation analysis for weakly singular Volterra integral equations with variable exponent

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Piecewise polynomial collocation of weakly singular Volterra integral equations (VIEs) of the second kind has been extensively studied in the literature, where integral kernels of the form $(t-s)^{-\alpha}$ for some constant $\alpha \in (0,1)$ are considered. Variable-order fractional-derivative differential equations currently attract much research interest, and in Zheng and Wang SIAM J. Numer. Anal. 2020 such a problem is transformed to a weakly singular VIE whose kernel has the above form with variable $\alpha = \alpha(t)$, then solved numerically by piecewise linear collocation, but it is unclear whether this analysis could be extended to more general problems or to polynomials of higher degree. In the present paper the general theory (existence, uniqueness, regularity of solutions) of variable-exponent weakly singular VIEs is developed, then used to underpin an analysis of collocation methods where piecewise polynomials of any degree can be used. The sharpness of the theoretical error bounds obtained for the collocation methods is demonstrated by numerical examples.

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A Numerical Method for Volterra-Fredholm Integral Equations in Two Dimensions

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We present an iterative numerical method for approximating solutions of two-dimensional Volterra-Fredholm integral equations of the second kind. As these equations arise in many applications, there is a constant need for accurate, but fast and simple to use numerical approximations to their solutions. The method proposed here uses successive approximations of Mann type and a suitable cubature formula. Mann's procedure is known to converge faster than the classical Picard iteration given by the contraction principle, thus yielding a better numerical method. The existence and uniqueness of the solution is derived under certain conditions. The convergence of the method is proved and error estimates for the approximations obtained are given. At the end, several numerical examples are analyzed, showing the applicability of the proposed method and good approximation results.

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Numerical solution of delay Volterra functional integral equations with variable bounds

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In this talk, we consider the numerical solution of Delay Volterra Integral Equations (DVIEs) represented by [1]

$$\sum_{k=0}^{m_1} P_k(x)y(\alpha_k x+\beta_k) = f(x) + \sum_{r=0}^{m_2} \lambda_r \int_{u_r(x)}^{v_r(x)} K_r(x,t)y(\mu_r t+\gamma_r)dt, \tag{17}$$

where $P_k(x)$, $f(x)$, $K_r(x,t)$, $u_r(x)$ and $v_r(x)$ are continuos functions on the interval $[a,b]$, $a \leq u_r(x) \leq v_r(x) \leq b$ and α_k, β_k, λ_k, μ_k and γ_k are appropriate constants.

In this work, a new type of orthogonal polynomials (as named Hahn polynomials [2]) are defined and applied to find the approximate solution of DVIEs (17). By applying the properties of these polynomials, explicit formulations for their integration and operational matrices are derived. By using the matrix-collocation method, a numerical approach is proposed to solve DVIEs (17). Some numerical results for several test problems are given to confirm the accuracy of this method.

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On the convergence of difference approximations to fractional differential problems in bounded domains

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Fractional differential equations are promising tools for characterizing anomalous diffusion in different fields and many interesting problems occur in bounded domains [1, 3]. When we add physical boundaries to fractional differential equations many difficulties arise and these include the development of accurate numerical approximations. Because the fractional derivative is nonlocal, the accuracy of its approximation near the boundary is strongly affected by the cut of the domain, that is, in general, the accuracy of the approximations of fractional order operators is lost near the boundary. To discretize these equations and in particular the fractional derivative we consider a known approximation that can be first order accurate when we have an open domain but it can be of lower order in bounded domains and sometimes not consistent [2]. Nevertheless, as we will show, the numerical methods are convergent and the first order rate of convergence can be recovered.

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Central part interpolation schemes for fractional differential equations

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A class of initial value problems for fractional integro-differential equations involving a Caputo fractional differential operator of order $\alpha \in (0, 1)$ is considered. First, the problem is reformulated as a weakly singular Volterra integral equation of the second kind. Then, a smoothing change of variables is used to improve the boundary behaviour of the exact solution of the underlying problem. After that, a collocation method based on central part interpolation by continuous piecewise polynomials on the uniform grid is constructed. The central part interpolation approach was introduced in [1] for solving Fredholm integral equations of the second kind and it has shown accuracy and numerical stability advantages compared to standard piecewise polynomial collocation methods, including collocation at Chebyshev knots [2]. In the present talk this approach is modified to solve fractional differential equations. The optimal convergence estimates are derived and the theoretical results are tested by some numerical experiments.

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S. 12 Special Functions and Applications

In this session we aim at gathering experts in a large range of aspects of special functions including applications. The topics include e.g. classical special functions such as hypergeometric functions, relations to orthogonal polynomials incl. Painleve equations and Riemann Hilbert problems, asymptotic analysis, approximation theory, q-special functions and connections to combinatorics, and special polynomials and functions used in fractional calculus.

Organizers:
Clemente Cesarano, Uninettuno University
Henrik Laurberg Pedersen, University of Copenhagen

A class of special functions using Fourier transforms of orthogonal polynomials on the unit disk

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The study of orthogonal polynomials and their transformations have been the subject of many papers during the last several years. By the Fourier transform or other integral transforms, some univariate orthogonal polynomials systems which are mapped onto each other have been studied in [5, 7]. Recently, there have been many papers on Fourier transforms of univariate and multivariate orthogonal polynomials [3, 4, 6, 8].
In this work, we present Fourier transform of multivariate orthogonal polynomials on the unit disk $B^r = \{\boldsymbol{x} \in \mathbb{R}^r : \|\boldsymbol{x}\| \leq 1\}$ (see [2]) and we write them in terms of continuous Hahn polynomials. By using the obtained Fourier transforms and Parseval's identity [1], we derive a new family of multivariate orthogonal functions.

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Fast and accurate evaluation of dual Bernstein polynomials

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Dual Bernstein polynomials have strong connections to hypergeometric functions and the shitfed Jacobi and Hahn families of orthogonal polynomials. They find many applications in approximation theory, computational mathematics, numerical analysis, and computer-aided geometric design. In this context, one of the main problems is fast and accurate evaluation both of these polynomials and their linear combinations. New simple recurrence relations of low order satisfied by dual Bernstein polynomials were given in [1] and expanded upon in [2]. In particular, a first-order non-homogeneous recurrence relation linking dual Bernstein and shifted Jacobi orthogonal polynomials has been obtained. When used properly, it allows to propose fast and numerically efficient algorithms for evaluating all $n+1$ dual Bernstein polynomials of degree n with $O(n)$ computational complexity.

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Electromagnetic Waves Confined in Annular Regions

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Electrodynamics configurations are obtained by solving the classical wave equations for the electric and the magnetic fields, in domains where the multiplicity of a given eigenvalue of the vector Laplacian is equal to four. This allows for the determination of interesting periodic solutions. The analysis is carried out in annular domains, and requires the determination of appropriate combinations of Bessel's functions and other special functions. Possible applications are in the field of plasma physics. In particular, we show how entrapped waves circulating in annular cavities may explain the stability properties of the phenomenon known as *ball lightning* [1].

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Computation of confluent hypergeometric functions and applications

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Confluent hypergeometric functions occur in many applications in applied mathematics, physics and engineering. Despite their importance, few algorithms are available for calculating any of the standard solutions of Kummer's equation in the case of real or complex parameters. In this talk, we present recent advances in the computation of the Kummer function U(a, b, x) [1]. As we will see, asymptotic expansions [2] are very important in the resulting algorithm. On the other hand, confluent hypergeometric functions play a key role in the asymptotic analysis of Fermi-Dirac integrals [3]. The evaluation of these integrals and their derivatives is necessary for various problems in applied and theoretical physics, such as stellar astrophysics, plasma physics or electronics. In this lecture we will show that the use of these expansions makes it possible to calculate the functions efficiently and with high accuracy for a large number of parameters [4].

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Generalized hypergeometric solutions of the Fuchsian linear differential equations

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We present infinitely many solutions of the general Heun equation in terms of the generalized hypergeometric functions ${}_{p+1}F_p$. Each solution assumes two restrictions imposed on the involved parameters: a characteristic exponent of a singularity should be a non-zero integer and the accessory parameter should obey a polynomial equation. [1],[2]

Next, we show that the single confluent Heun equation with non-zero ε (this is the parameter characterizing the irregular singularity at the infinity) admits infinitely many solutions in terms of the generalized hypergeometric functions ${}_pF_p$. For each of these solutions a characteristic exponent of a regular singularity of the confluent Heun equation is a non-zero integer and the accessory parameter obeys a polynomial equation. Each solution can be written as a linear combination with constant coefficients of a finite number of the Kummer confluent hypergeometric functions. [3]

Furthermore, we show that for the Ince limit $\varepsilon = 0$ the confluent Heun equation admits infinitely many solutions in terms of the functions ${}_pF_{p+1}$. Here again a characteristic exponent of a regular singularity should be a non-zero integer and the accessory parameter should obey a polynomial equation. This time, each solution can be written as a linear combination with constant coefficients of a finite number of the Bessel functions. [3]

Finally, we show that a Fuchsian differential equation having five regular singular points admits solutions in terms of a single generalized hypergeometric function for infinitely many particular choices of equation parameters. Each solution assumes four restrictions imposed on the parameters: two of the singularities should have non-zero integer characteristic exponents and the accessory parameters should obey polynomial equations. [4]

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New results on the Lambert W function

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A number of new results on the Lambert W function [1] and applications are presented. The asymptotic behaviour of the non-principal branches $W_k(z), k \neq 0$ around $z = 0$ has been overlooked until now; the expansions are now detailed. The asymptotic expansions at $|z| \to \infty$ are known to be convergent for large z. It is now possible to give the boundary in the complex plane of the convergence domain. Various expressions containing W are Stieltjes functions [2]; direct proofs are given. Also, there are applications in statistics of the $k = -1$ branch, and these applications lead to Stieltjes functions there also.

Recently, a proposal was presented regarding alternative branch structures for W [3]. This proposal will be discussed.

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Generalised higher order Freud polynomials

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Gabor Szegő pioneered much of what is known in the theory of orthogonal polynomials on finite intervals but did not carry his ideas over to infinite intervals, despite there being significant differences. In the second half of the 20th century, starting with the work of Géza Freud on orthogonal polynomials on $\mathbb{R}$, the study of Freud-type polynomials and their generalisations flourished. In this talk I will discuss symmetric semi-classical polynomials orthogonal with respect to generalisations of higher order Freud weights.

On Apostol-type polynomials

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The generating functions for the special polynomials are important from different view points and help in finding connection formulas, recursive relations, difference equations, and in solving problems in combinatorics and encoding their solutions. In this talk, a new class of the degenerate Apostol–type Hermite polynomials is introduced. Certain algebraic and differential properties of there polynomials are derived. Most of the results are proved by using generating function methods. (for example [1],[2], [3], [4] and [5]).

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On the second–order holonomic equation for Sobolev–type orthogonal polynomials

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In this talk it is considered a general approach to the study of orthogonal polynomials related to Sobolev inner products which are defined in terms of divided-difference operators having the fundamental property of leaving a polynomial of degree $n-1$ when applied to a polynomial of degree n. The main focus is on the analytic properties for the orthogonal polynomials, including the second-order holonomic difference equation satisfied by them. (This talk is based on the paper Rebocho, Maria das Neves, On the second-order holonomic equation for Sobolev-type orthogonal polynomials. Appl. Anal. 101 (2022), no. 1, 314–336.)

S13. Theoretical aspects of Isogeometric Analysis and recent applications

Isogeometric analysis (IgA) is a method for the numerical simulation of problems governed by partial differential equations. One of the key points of IgA is the retainment of the description of the domain where the PDE is defined as given by a CAD system (so in terms of B-splines, NURBS or their generalizations), instead of approximating it by a triangular/polygonal mesh. Indeed the term ”Isogeometric” is due to the fact that the solution space for dependent variables is represented in terms of the same functions which describe the geometry (i.e. splines). Other good features are the gain of high flexibility in the smoothness of the discretization space, and the simplification of mesh refinement by eliminating the need for communication with the CAD geometry once the initial mesh is constructed. The research on IgA has been oriented into two main directions. On one hand to apply the available CAGD techniques to different PDEs, ranging from fluids, structures, phase-field modeling, electromagnetics, shape and topology optimization, till discrete and diffuse modeling of crack propagation. On the other hand to develop new and more flexible representations like hierarchical splines, generalized Tchebychev splines, and locally-refinable B-splines and to investigate on the related theoretical issues. Another important aspect is also the consistent treatment of trimmed patches and multi-patch geometry. The purpose of this special session is to give an overview on several theoretical and applicative aspects of IgA recently arisen.

Organizers:

Alessandra Aimi, University of Parma
Maria Lucia Sampoli, University of Siena
Alessandra Sestini, University of Florence

IGA-Energetic BEM for the numerical solution of 2D wave scattering problems in the space-time domain

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The Energetic Boundary Element Method (BEM) was proposed in [1] for the numerical solution of 2D exterior wave propagation problems. The hyperbolic differential model is transformed, using the fundamental solution of the wave operator, into a Boundary Integral Equation (BIE), which is then written into an energy-dependent weak form and discretized by a Galerkin-type approach. Compared to other space-time discretizations of the wave equation [4], the energetic weak form offers desirable accuracy and stability properties [1].

Taking into account the model problem of 2D soft scattering of acoustic waves by open arcs represented by B-spline (or NURBS) curves, in this talk we discuss some recent advances in the coupling of Energetic BEM with the powerful Isogeometric Analysis (IGA) approach for what concerns discretization in space variables. Indeed, IGA, proposed by T. J. R. Hughes and collaborators [5] in the context of the FEM to "bridge the gap" between design and analysis using B-splines and NURBS as shape functions, naturally fits into BEMs, allowing an exact representation of curvilinear boundaries.

In this contribution, based on [2], numerical issues for an efficient integration of the singular kernel related to the fundamental solution of the wave operator and dependent on the propagation wavefront, will be described, highlighting the new challenges posed by the presence of curvilinear boundaries. Extensive numerical experiments will show, from a numerical point of view, convergence and accuracy of the proposed method as well as the superiority of IGA-Energetic BEM compared to the standard version of the method, based on lagrangian shape functions. Simulations on long time intervals allow to observe the consistency of the proposed method with the stationary IGA-BEM [3].

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Preconditioners for adaptive spaces or spaces for preconditioners?

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Adaptive methods for the numerical solution of PDEs require constructions of discrete spaces in which the resolution varies in the domain of interest. In IGA this has been achieved by breaking the global tensor product structure of multivariate splines as shown by many different spaces such as hierarchical splines and T-spline among others. The available spaces have the desired approximation properties, but they require new or adapted preconditioning techniques, examples for the mentioned spaces can be found in [1, 2, 3].

The talk will present a different take at the construction of the discretization space going backward: from preconditioning techniques for Krylov methods such as fast-diagonalization[4] and subspace-correction[5] to a discretization space for PDEs.

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Efficient Quadrature in Isogeometric Galerkin methods and Isogeometric Boundary Element methods

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We present the construction of quadrature rules for the approximation of integrals that occur when elliptic problems are numerically solved using Isogeometric Analysis (IgA) both in the standard Galerkin and in the Galerkin Boundary Element frameworks.

In particular the IgA Galerkin discretization askes for the computation of integrals involving splines, and a new formation and assembly strategy was proposed which resulted in significant speedups in the formation and assembly time of the Galerkin mass matrix, see [3]. Moreover, recurrence relations can be used in order to require exactness of the rules also in cases where singularity occurs, as in the case of BEMs [1, 2].

Moreover, we discuss various important details for the practical implementation of the quadrature formation strategies proposed in [1, 2, 3, 4]. Specifically, we discuss the weighted quadrature scheme to accurately integrate the elements of the stiffness matrix and we discuss efficient assembly in the BEM case where singular integrals appear. We will review such approaches and then focus on the use of spline quasi-interpolation to approximate integrand factors, that gives a final formulation where integrals can be evaluated evaluated via recurrence relations, as proposed recently in [4]. Considered cases include hypersingular and singular integrands. Convergence results of the proposed quadrature rules are given, with respect to both smooth and non smooth integrands. Numerical tests confirm the behavior predicted by the analysis.

We show that the accuracy is maintained while the computational burden of forming the matrix equations is significantly reduced.

The research is a part of collaborations with A. Aimi, A. Falini, M.L. Sampoli, A. Sestini, G. Sangalli, M. Tani.

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A multi-patch IgA-BEM model for 3D Helmholtz problems

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An Isogeometric Boundary Element Method (IgA-BEM) is considered for the numerical solution of Helmholtz problems on 3D domains admitting a smooth conformal multi-patch representation of the boundary surface. In particular, the resulting Boundary Integral Equations are discretized by a classical collocation method and a spline based quasi-interpolation quadrature scheme is developed for both regular and singular kernels. Suitable spline-product spaces are constructed locally on the support of every B-spline basis function and in case of singular (or nearly singular) integrals, a singularity extraction technique is combined with a an elegant recursive formula, which is based on the analytical evaluation of fundamental polynomial moments. Taking advantage of the local construction on each basis support, the matrix assembly phase can be efficiently carried out by using a function-by-function approach. Relevant benchmarks show that the expected convergence orders are achieved and good accuracy is reached by using a small number of quadrature nodes.

Spectral Analysis of Isogeometric Immersed Discretizations

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We consider the tensor-product B-spline isogeometric analysis discretization of a variable- coefficient symmetric elliptic problem. The isogeometric discretization is coupled with an immersed boundary (embedded domain) method that preserves the symmetry of the prob- lem [1, 5].

We present a spectral analysis of the matrices resulting from this discretization [3]. In particular, our interest is focused on the asymptotic distribution of the eigenvalues as the mesh-fineness parameter n tends to ∞, i.e., as the mesh is progressively refined to get increasingly accurate approximations. Such analysis plays a role both in the design of efficient solvers for the resulting linear systems and in the study of the accuracy with which of the proposed discretization method approximates the spectrum of the differential operator underlying the considered elliptic problem.

The spectral analysis tools we use are entirely based on the theory of (reduced) generalized locally Toeplitz (GLT) sequences [2, 4], which is introduced in the talk along with the obtained spectral results.

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C^1 hierarchical spline constructions on planar multi-patch geometries for adaptive IGA

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We present an adaptive isogeometric method for the numerical approximation of partial differential equations defined on certain planar multi-patch geometries with C^1 hierarchical splines. We first discuss key properties of the considered hierarchical spline space and its associated basis, such as nestedness on refined meshes and, under a mild assumption on the mesh near the vertices, linear independence of the basis. We then present a refinement algorithm with linear complexity, which guarantees the construction of graded hierarchical meshes that fulfill the condition for linear independence. A selection of numerical examples will confirm the potential of the adaptive scheme on different multi-patch domains.

Optimal spline spaces are outlier free

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Smooth isogeometric methods obtain an extremely good spectral approximation of differential operators when compared to classical finite element methods, however, they still present a few poorly approximated eigenvalues and eigenfunctions which are referred to as outliers. This superior spectral behavior translate into improved numerical simulations, especially for explicit dynamics, but the presence of outliers dampens the possible gain. In this talk we explain how isogeometric discretizations using the optimal spline spaces identified in [1] lead to outlier free approximations of eigenvalue problems. This talk is based on the results of [2, 3].

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Smooth splines on unstructured meshes

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Isogeometric Analysis [1] generalizes classical finite element analysis and, at the same time, intends to seamlessly unify it with the field of Computer-Aided Design. Achieving this latter objective would encapsulate the entire engineering design-through-analysis workflow in a uniform framework, yielding a significant boost to the efficiency of current engineering workflows. A central problem in achieving this objective is design and analysis of complex two and three dimensional geometries of arbitrary topologies. This requires moving beyond splines on structured quadrilateral and hexahedral meshes – globally structured meshes cannot be used to represent arbitrary geometries and parameterization singularities (i.e., extraordinary points, polar points and extraordinary edges) must be introduced. Thus, the design and analysis of complex geometries requires that we construct and study spaces of smooth splines on unstructured meshes. This talk will present an overview of recently proposed analysis-suitable spline constructions (e.g., [2, 3]) as well as their generalizations [4].

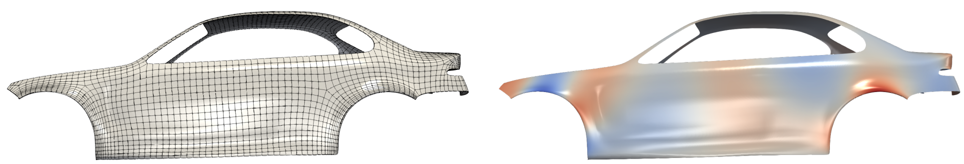

Figure: An unstructured spline representation of a car body (left) and its free vibration analysis (right).

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Collocation Isogeometric Approximation of acoustic wave problems

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In this presentation we consider the numerical approximation of acoustic wave problems with absorbing boundary conditions by the Isogeometric discretization in space, and Newmark scheme in time, both explicit and implicit [2, 5]. Isogeometric Analysis (IGA) allows not only the standard p- and hp-refinement of hp- finite elements and spectral elements, where p is the polynomial degree of the C^0 piecewise polynomial basis functions, but also a novel k-refinement where the global regularity k of the IGA basis functions is increased proportionally to the degree p, up to the maximal IGA regularity $k = p - 1$ [2].
In the framework of NURBS-based IGA, first we have considered Galerkin approaches [4] and then we have moved on to collocation methods, that in general optimize the computational cost, still taking advantage of IGA geometrical flexibility and accuracy [1, 3].
Proper boundary conditions are also considered. While homogeneous Neumann conditions provide a good mathematical representation of a free surface, absorbing boundary conditions are imposed in order to simulate wave propagation in infinite domains, by truncating the original unbounded region into a finite one.
Several numerical examples illustrate the stability and convergence properties of the numerical collocation IGA methods with respect to all the IGA approximation parameters, namely the local polynomial degree p, regularity k, mesh size h, and to the time step size Δt of the Newmark schemes [5]. Some numerical results on the spectral properties of the Collocation IGA mass and stiffness matrices are also presented.

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Poster Session

Organizers:
Maria Rosaria Capobianco, CNR, IAC
Maria Carmela De Bonis, University of Basilicata
Vita Leonessa, University of Basilicata

Triangular spline quasi-interpolants and their application in terrain modelling

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Digital Elevation Models (DEMs for short) have a wide application in Hydrology, Geology, Environment and Civil Engineering, among other disciplines. Sometimes there exists more than one DEM from the same area. The difference among them may be due to the fact that they come from different producers, different methodologies to capture them or different cell sizes used in their production. Many applications require that all DEMs used have the same cell size to be interoperable [1, 2, 3], which implies a resampling of the original DEM. This resampling can influence the quality of the final product, so the resampling method used very important aspect to take into account.

In this work we propose the construction of a triangular spline quasi-interpolant over the type-1 triangulation of the partition into squares associated with the dataset to be approximated. The triangulation is endowed with a Powell-Sabin 6-split. Instead of expressing the quasi-interpolant in terms of a basis of B-spline-like functions, it is constructed by directly setting the coefficients of the Bernstein-Bézier representation of its restriction to each of the micro-triangles into which each macro-triangle is decomposed. Each coefficient will be determined from the values to be approximated at the points of a neighbourhood of the micro-triangle under consideration, making use of rules that will guarantee the required regularity and order of approximation.

This quasi-interpolating spline will provide a new resampling method that will allow to study its quality when going from a higher resolution to a lower resolution.

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Perturbed iterative approximation for the solution of delay differential equations

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The purpose of this work is to obtain an approximation to the solution of a delay differential equation by combining a one–step Picard-type scheme and the use of Schauder bases in a suitable Banach space. The effectiveness of the proposed method is showed with some examples.

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FSK-PSK data processing based on direct approximation of the Hilbert transform

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We describe a signal processing method for demodulation of digital signals based on Hilbert transform (HT).
We review the signal processing theory and the method of Analytic Signal transformation (AS) and their algorithms which are implemented by FFT, then we propose a direct method for the numerical approximation of the Hilbert transform that is a generalization of the algorithm presented in [1].
The proposed algorithm provides the estimate of instantaneous frequency and phase of the received signals, and can be used for both binary communication based on phased-shifting keying (PSK) and frequency-shifting keying (BFSK) [2].
Typical applications include data analysis as a bank of matched filters [3], data communication of electric and acoustic soil response and sea autonomous platforms.

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Filtered product integration rules for the Hilbert Transform in $(-1,1)$

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We present a numerical method for approximating Hilbert transforms of the type

$$\mathcal{H}^u f(t) = \int_{-1}^{1} \frac{f(x)}{x-t} u(x)dx = \lim_{\epsilon \to 0} \int_{|x-t|\geq \epsilon} \frac{f(x)}{x-t} u(x)dx, \qquad -1 < t < 1,$$

where $u(x) = v^{\gamma,\delta}(x) := (1-x)^\gamma (1+x)^\delta$, $\gamma, \delta > -1$ is a Jacobi weight.

Fixed another Jacobi weight $w(x) = v^{\alpha,\beta}(x)$, and denoting by $\{p_j(w)\}_j$ the corresponding orthonormal polynomial sequence, fixed two integers n, m with $0 < m < n$, the $n-$th filtered de la Vallée Poussin (VP) polynomial of f is defined as [1]:

$$V_n^m(w, f, x) = \sum_{k=1}^{n} f(x_k) \Phi_{n,k}^m(x),$$

where $\{\Phi_{n,k}^m\}_{k=1:n}$ are the fundamental VP polynomials

$$\Phi_{n,k}^m(x) = \lambda_{n,k}(w) \sum_{j=0}^{n+m-1} \mu_{n,j}^m p_j(w,x) p_j(w, x_k),$$

$\lambda_{n,k}(w)$ and x_k being the Cristhoffel numbers and the zeros of $p_n(w)$ respectively and $\mu_{n,j}^m$ are the following filters

$$\mu_{n,j}^m := \begin{cases} 1 & \text{if} \quad j = 0, \ldots, n-m, \\ \dfrac{n+m-j}{2m} & \text{if} \quad n-m < j < n+m. \end{cases}.$$

The new product type quadrature rule has been obtained by approximating f by $V_n^m(w, f)$ [2]. The convergence and stability are studied in suitable Besov type spaces. A comparison with the product quadrature rule based on the approximation of f by the Lagrange polynomial interpolating f at the same zeros $\{x_k\}_{k=1}^n$ of $p_n(w)$ is also proposed [3].

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Numerical simulation of a multi-group age-of-infection model

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Age of infection epidemic models [1, 3], based on non-linear integro-differential equations, naturally describe the evolution of diseases whose infectivity depends on the time since becoming infected. Here we consider a multi-group age of infection model [2] and we extend the investigations in [4], [5] and [6] to provide numerical solutions that retain the main properties of the continuous system. In particular, we use Direct Quadrature methods and prove that the numerical solution is positive and bounded. Furthermore, in order to study the asymptotic behavior of the numerical solution, we formulate discrete equivalents of the final size relation and of the basic reproduction number and we prove that they converge to the continuous ones, as the step-size of the discretization goes to zero.

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